

# LOWER BOUNDS FOR BRUSS' ODDS PROBLEM WITH MULTIPLE STOPPINGS

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We give asymptotic lower bounds of the value for Bruss' optimal stopping problem with multiple stopping chances. It interestingly consists of the asymptotic threshold values in the optimal multiple stopping strategy. Another interesting implication of the result is that the asymptotic value for each secretary problem with multiple stoppings is in fact a typical lower bound in a much more general class of multiple stopping problems as modifications of odds problem.

**1. Introduction.** We provide asymptotic lower bounds of probability of “win” (*i.e.*, obtaining the last success) for odds problem with multiple stoppings, which has some general setting in optimal stopping theory. The problem may be stated as follows. We observe sequentially a sequence of independent 0/1 random variables, that is, Bernoulli sequence,  $X_1, X_2, \dots, X_N$ , where  $N$  is a given positive integer and the distribution is  $\Pr[X_i = 1] = p_i$ ,  $\Pr[X_i = 0] = 1 - p_i = q_i$ ,  $0 < p_i < 1$  for each  $i$ . We say “success” if  $X_i = 1$  and “failure” if  $X_i = 0$ . We want to stop on the last success with multiple stopping chances.

This is an attractive problem setting. We may quote from Bruss [6]: “Many stopping problems are of a similar kind. One often wants to stop on the very last success. For instance, investors are typically interested stopping on the last success in a given period, where a success is a price increase in a long position and a decrease in a short position. Similarly, venture capital investors often try to put all reserved capital in the last technological innovation in the targeted field. In secretary problems, we want to select the best candidate (which means stopping on the last record value) and so on.”

For single stopping problem, it has an elegant and simple optimal stopping strategy known as *Odds theorem* or *Sum the Odds theorem*. For the odds problem, a typical lower bound for an asymptotic optimal value (probability of win), when  $N$  goes to infinity, is shown to be  $1/e$  in Bruss [6]. The value often appears in the literature of the many modifications of secretary prob-

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<sup>\*</sup>This work was supported by KAKENHI (23651157).

AMS 2000 subject classifications: Primary 60G40

Keywords and phrases: optimal stopping, odds problem, lower bounds, multiple stopping

lem having a specified probability of success,  $p_i = 1/i$ , [See, e.g., Pfeifer [18], Samuels [19] for a review and others.], and in the one of the variations of prophet inequality based on relative ranks, [See, e.g., Hill and Krengel [16] and Hill and Kennedy [15].]. The value  $1/e$  also appears in the asymptotic threshold value of the optimal stopping strategy for secretary problem. For large  $N$ , the optimal strategy is to pass the all candidate until  $(1/e) \times N$  and then stop at the first relative best (if any) thereafter. Another variations of odds problem are studied by Hsiau and Yang [17] for Markov-dependent trials with single stopping, Ano, Kakie and Miyoshi [2] for Markov-dependent trials with multiple stoppings, Tamaki [21] for stopping on any of the last  $m$  successes, and Bruss and Louchard [8] for unknown success probability.

For each multiple stopping odds problem, the questions arise; (a) what is the optimal multiple stopping strategy to maximize the probability of win? (b) what is the maximum probability of win and the lower bound? (c) what is the asymptotic lower bound of probability of win for any sequence  $\{p_1, p_2, \dots, p_N\}$  of the success probability and for any fixed number of stopping chances, when  $N$  goes to infinity? (d) is the lower bound for each fixed number of stopping chances attained by the corresponding one of the secretary problem with multiple stopping chances? In other words, does secretary problem still keep benchmark position of the bound for odds problem? (f) if so, is the lower bound, strange to say, composed of the asymptotic threshold values in the optimal multiple stopping strategy? The first answer has been provided in Ano, Kakinuma and Miyoshi [3]. When the decision maker has more  $m$  stopping chances, there always exist threshold values satisfying inequalities  $1 \leq i^{(m)} \leq i^{(m-1)} \leq \dots \leq i^{(1)} \leq N$ , and for each  $\ell \in \{1, 2, \dots, m\}$  the optimal stopping strategy is given by

$$\tau^{(\ell)} = \min\{i \mid i \geq \max\{i^{(\ell)}, \tau^{(\ell+1)} + 1\} \text{ and } X_i = 1\},$$

where  $\tau^{(m+1)}=0$ . The second and third questions are partially answered in also [3].

Let us summarize main results of this paper as follows. (1) We give the probability of win for odds problem with  $m$ -stoppings. (2) For any sequence  $\{p_1, p_2, \dots, p_N\}$  of the success probability satisfying some conditions and for any fixed number  $m$  of stopping chances, we give the lower bound of probability of win for odds problem with  $m$ -stoppings. An efficient algorithm to calculate lower bounds is presented. (3) We show that the asymptotic lower bound of probability of win for odds problem with  $m$ -stoppings is attained by the asymptotic maximum probability of win for secretary problem with  $m$ -stoppings. This answers the question (c). So that, the answer for (d) is “yes.” (4) Finally, we show a beautiful connection between threshold val-

ues of optimal stopping strategy and probability of win. The asymptotic lower bound is composed of the asymptotic threshold values,  $\{i_N^{(m)}, i_N^{(m-1)}, \dots, i_N^{(1)}\} = \{i^{(m)}, i^{(m-1)}, \dots, i^{(1)}\}$ , in the optimal multiple stopping strategy, and given by

$$(1.1) \quad \lim_{N \rightarrow \infty} \sum_{k=1}^m \frac{i_N^{(k)}}{N} = \lim_{N \rightarrow \infty} P_N^{(\text{win})}(m)$$

where  $P_N^{(\text{win})}(m)$  denotes a corresponding probability of win for odds problem (and also for secretary problem) with  $m$ -stoppings.

For example, when  $m = 1$ , it is well-known that the asymptotic threshold value  $e^{-1}$  is equals to the asymptotic probability of win. When  $m = 2$ , the asymptotic probability of win for odds problem is  $e^{-1} + e^{-3/2}$  that equal to the one for secretary problem. Optimal threshold strategy for odds problems with 2-stoppings is a threshold strategy,  $\{i_N^{(2)}, i_N^{(1)}\}$ . Corresponding asymptotic thresholds values are  $\lim_{N \rightarrow \infty} (i_N^{(2)}/N) = e^{-3/2}$  and  $\lim_{N \rightarrow \infty} (i_N^{(1)}/N) = e^{-1}$ . These values are equals to the ones for secretary problem with 2-stoppings. When  $m = 3$ , asymptotic probability of win for odds problem is  $e^{-1} + e^{-3/2} + e^{-47/24}$ . When  $m = 4$ , the corresponding one is  $e^{-1} + e^{-3/2} + e^{-47/24} + e^{-2761/1152}$  that equals to the one for secretary problem. The connection between the probability of win and the threshold values remains in a similar way as the case of 2-stoppings. In secretary problem, multiple stopping setting may go back to Gilbert and Mosteller [13]. Exact these values,  $e^{-1}, e^{-3/2}, e^{-47/24}$  and  $e^{-2761/1152}$  for  $m = 1, 2, 3$  and 4 were founded in Bruss [5]. We give the values for  $m = 5, 6, 7, 8, 9$  and 10 in Table 2.

**2. Win Probability of Threshold Strategy.** Let  $X_1, X_2, \dots, X_N$  be a sequence of independent 0/1 random variables. For any  $i \in \{1, 2, \dots, N\}$ , we denote  $p_i = E[X_i]$ . Throughout this paper, we assume that  $0 < p_i < 1$  for any  $i$ . We denote a probability of failure  $1 - p_i$  by  $q_i$  and an odds  $\frac{p_i}{q_i}$  of  $X_i$  by  $r_i$ .

In this section, we introduce a threshold strategy for odds problem with  $m$ -stoppings defined on  $X_1, X_2, \dots, X_N$ . A threshold strategy is defined by a vector of integer *threshold values*  $\mathbf{i} = (i^{(m)}, i^{(m-1)}, \dots, i^{(1)})$  satisfying inequalities  $1 \leq i^{(m)} \leq i^{(m-1)} \leq \dots \leq i^{(1)} \leq N$ . Simply put, a threshold strategy  $\mathbf{Threshold}(i^{(m)}, i^{(m-1)}, \dots, i^{(1)})$  observes a random variable at each iteration and selects a random variable of success if and only if the number of previously selected variables is less than the number of passed threshold values on and before the iteration. In the following, we give a precise

definition. A variable **slack** denotes a remained capacity for selection. The initial value of **slack** is equal to 0. At  $i$ -th iteration, we increase **slack** by  $|\{k \in \{1, 2, \dots, m\} \mid i = i^{(k)}\}|$ , which is the number of thresholds equal to  $i$ . When we select a random variable, we decrease **slack** by 1. If we have selected the last success before  $i$ -th iteration, we set **result** to **win**. Else, we set **result** to **lose**. When the value of **result** is **win**, variable **last** denotes the index of last success before the iteration.

**Threshold** $(i^{(m)}, i^{(m-1)}, \dots, i^{(1)})$

**Step 0:** Set  $i := 0$ ; **slack** := 0; **result** := **lose**; **last** :=  $\emptyset$ .

**Step 1:** Set  $i := i + 1$ ; **slack** := **slack** +  $|\{k \in \{1, 2, \dots, m\} \mid i = i^{(k)}\}|$ .

Set  $x_i := \begin{cases} 1 & (\text{with probability } p_i), \\ 0 & (\text{with probability } q_i). \end{cases}$

**Step 2:** If  $[x_i = 0]$ , then goto Step 3.

Else if  $[x_i = 1 \text{ and } \mathbf{slack} > 0]$

then set **slack** := **slack** - 1; **last** :=  $\{i\}$ ; **result** := **win**  
(and we say that index  $i$  is *accepted*).

Else (both  $x_i = 1$  and **slack** = 0 hold),

set **last** :=  $\emptyset$ ; **result** := **lose**

(and we say that index  $i$  is *rejected*).

**Step 3:** If  $[i < N]$ , then goto Step 1.

Else if  $[i = N \text{ and } \mathbf{result} = \mathbf{win}]$ , then output the index in **last** and stop.

Else output “lose” and stop.

If the above procedure outputs an index  $i$ , then we obtained the last success attained by  $X_i$ .

In the rest of this section, we discuss the probability of win of a threshold strategy **Threshold** $(i^{(m)}, i^{(m-1)}, \dots, i^{(1)})$ . If an index (corresponding to the last success) is obtained by executing **Threshold** $(i)$ , we say that a vector of realized values  $(x_1, x_2, \dots, x_N) \in \{0, 1\}^N$  of  $(X_1, X_2, \dots, X_N)$  is *winning*. We introduce a partition  $B_1, B_2, \dots, B_{m+1}$  of index set  $\{1, 2, \dots, N\}$  defined by

$$B_k = \begin{cases} \{i \in \{1, 2, \dots, N\} \mid i^{(1)} \leq i \leq N\} & (k = 1), \\ \{i \in \{1, 2, \dots, N\} \mid i^{(k)} \leq i < i^{(k-1)}\} & (1 < k \leq m), \\ \{i \in \{1, 2, \dots, N\} \mid 1 \leq i < i^{(m)}\} & (k = m + 1). \end{cases}$$

For any  $k$ , an index set  $B_k$  is called a *block*. Given a 0-1 vector  $\mathbf{x} \in \{0, 1\}^N$ ,  $\mathbf{b}(\mathbf{x}) = (b_m, b_{m-1}, \dots, b_1)$  denotes an  $m$ -dimensional vector, called a *pattern vector* of  $\mathbf{x}$ , satisfying  $b_k = \sum_{i \in B_k} x_i$  ( $k \in \{1, 2, \dots, m\}$ ). Here we note that elements of vector  $\mathbf{b}(\mathbf{x})$  are arranged in decreasing order of indices.

For any vector  $\mathbf{b} = (b_d, b_{d-1}, \dots, b_1)$ , we say that a vector  $(b_{d'}, b_{d'-1}, \dots, b_1)$  satisfying  $d \geq d' \geq 1$  is a *left truncated subvector* of  $\mathbf{b}$ . Throughout this paper,  $\mathbf{Z}_+$  denotes a set of non-negative integers.

When we consider a single-stopping problem (discussed in [7]), a vector  $\mathbf{x} \in \{0, 1\}^N$  is winning if and only if its pattern vector  $\mathbf{b}(\mathbf{x})$  satisfies  $b_1 = 1$ , i.e., one-dimensional vector  $(b_1) = (1)$  is a left truncated subvector of  $\mathbf{b}(\mathbf{x})$ . The probability of win of a threshold strategy  $\text{Threshold}(i^{(1)})$  is equal to

$$\left( \prod_{i \in B_1} q_i \right) \left( \sum_{i \in B_1} r_i \right).$$

Next, we consider a case that  $m = 2$  (discussed in [3]). We assume that  $|B_1| \geq 2$ . It is easy to show that a vector  $\mathbf{x} \in \{0, 1\}^N$  is winning if and only if its pattern vector  $\mathbf{b}(\mathbf{x})$  has a left truncated subvector contained in a set  $\{(1), (1, 0), (0, 2)\}$ . Since every integer vector has at most one left truncated subvector in the set  $\{(1), (1, 0), (0, 2)\}$ , the probability of win of  $\text{Threshold}(i^{(2)}, i^{(1)})$  is equal to

$$\left( \prod_{i \in B_1} q_i \right) \left( \sum_{i \in B_1} r_i \right) + \left( \prod_{i \in B_2 \cup B_1} q_i \right) \left( \sum_{i \in B_2} r_i + \sum_{\{i, i'\} \subseteq B_1} r_i r_{i'} \right).$$

Now we discuss a general case. First, we show a necessary and sufficient condition that a vector  $\mathbf{x} \in \{0, 1\}^N$  becomes a winning vector of  $m$ -stopping problem. We define a set of  $k$ -vectors

$$\widehat{\Xi}_k = \left\{ (b_k, b_{k-1}, \dots, b_1) \in \mathbf{Z}_+^k \left| \begin{array}{lcl} 1 & \geq & b_k, \\ 2 & \geq & b_k + b_{k-1}, \\ & \vdots & \\ k-1 & \geq & b_k + b_{k-1} + \dots + b_2, \\ k & \geq & b_k + b_{k-1} + \dots + b_2 + b_1 \geq 1 \end{array} \right. \right\}$$

for each  $k$  in  $\{1, 2, \dots, m\}$ . The following lists give vectors in  $\widehat{\Xi}_1, \widehat{\Xi}_2, \dots, \widehat{\Xi}_4$ ;

$$\begin{aligned}
\widehat{\Xi}_1 &= \{(1)\}, \\
\widehat{\Xi}_2 &= \{(1, 1), (1, 0), (0, 2), (0, 1)\}, \\
\widehat{\Xi}_3 &= \left\{ \begin{array}{l} (1, 1, 1), (1, 1, 0), (1, 0, 2), (1, 0, 1), (1, 0, 0), \\ (0, 2, 1), (0, 2, 0), (0, 1, 2), (0, 1, 1), (0, 1, 0), \\ (0, 0, 3), (0, 0, 2), (0, 0, 1) \end{array} \right\}, \\
\widehat{\Xi}_4 &= \left\{ \begin{array}{l} (1, 1, 1, 1), (1, 1, 1, 0), (1, 1, 0, 2), (1, 1, 0, 1), (1, 1, 0, 0), \\ (1, 0, 2, 1), (1, 0, 2, 0), (1, 0, 1, 2), (1, 0, 1, 1), (1, 0, 1, 0), \\ (1, 0, 0, 3), (1, 0, 0, 2), (1, 0, 0, 1), (1, 0, 0, 0), (0, 2, 1, 1), \\ (0, 2, 1, 0), (0, 2, 0, 2), (0, 2, 0, 1), (0, 2, 0, 0), (0, 1, 2, 1), \\ (0, 1, 2, 0), (0, 1, 1, 2), (0, 1, 1, 1), (0, 1, 1, 0), (0, 1, 0, 3), \\ (0, 1, 0, 2), (0, 1, 0, 1), (0, 1, 0, 0), (0, 0, 3, 1), (0, 0, 3, 0), \\ (0, 0, 2, 2), (0, 0, 2, 1), (0, 0, 2, 0), (0, 0, 1, 3), (0, 0, 1, 2), \\ (0, 0, 1, 1), (0, 0, 1, 0), (0, 0, 0, 4), (0, 0, 0, 3), (0, 0, 0, 2), (0, 0, 0, 1) \end{array} \right\}.
\end{aligned}$$

LEMMA 2.1. *A vector  $\mathbf{x} \in \{0, 1\}^N$  is a winning vector of  $m$ -stopping problem if and only if there exists an integer  $k \in \{1, 2, \dots, m\}$  satisfying that a pattern vector  $\mathbf{b}(\mathbf{x}) = (b_m, b_{m-1}, \dots, b_1)$  has a left truncated subvector  $(b_k, b_{k-1}, \dots, b_1) \in \widehat{\Xi}_k$ .*

PROOF. Assume that  $\mathbf{x} \in \{0, 1\}^N$  satisfies that whose pattern vector  $\mathbf{b}(\mathbf{x}) = (b_m, b_{m-1}, \dots, b_1)$  has a left truncated subvector in  $\widehat{\Xi}_k$ . Consider a case that we executed threshold strategy **Threshold**( $\mathbf{i}$ ) and  $\mathbf{x}$  was a vector of realized values of random variables  $(X_1, X_2, \dots, X_N)$ . Condition  $b_k + b_{k+1} + \dots + b_1 \geq 1$  in the definition of  $\widehat{\Xi}_k$  implies that there exists at least one index  $i$  with  $x_i = 1$  satisfying  $i^{(k)} \leq i$ . Other conditions of  $\widehat{\Xi}_k$  implies that at every iteration later than or equal to  $i^{(k)}$  of **Threshold**( $\mathbf{i}$ ), every index  $i'$  satisfying  $x_{i'} = 1$  is accepted. Thus, **Threshold**( $\mathbf{i}$ ) outputs the index of last success. From the above,  $\mathbf{x}$  becomes a winning vector.

Next, we discuss the inverse implication. Assume that  $\mathbf{x}$  is a winning vector. Consider a case that  $\mathbf{x}$  is a vector of realized values of  $(X_1, X_2, \dots, X_N)$  obtained in procedure **Threshold**( $\mathbf{i}$ ). From the assumption, **Threshold**( $\mathbf{i}$ ) outputs the index of last success. If no index is rejected by **Threshold**( $\mathbf{i}$ ), we obtain a desired result by setting  $k = m$ . Consider a case that **Threshold**( $\mathbf{i}$ ) rejected at least one index. Let  $B_{k'}$  be a block including rejected index and whose subscript is minimum. Since  $\mathbf{x}$  is winning, block  $B_{k'}$  does not includes the index of last success and thus  $k' \geq 2$ . Then  $\widehat{\Xi}_{k'-1}$  includes a left truncated subvector of the pattern vector of  $\mathbf{x}$ .  $\square$

For any  $k \in \{1, 2, \dots, m\}$ , we define

$$\Xi_k = \left\{ (b_k, \dots, b_1) \in \widehat{\Xi}_k \mid \begin{array}{l} k > \forall k' \geq 1, \forall \mathbf{b}' \in \widehat{\Xi}_{k'}, \\ \mathbf{b}' \text{ is not a left truncated subvector of } \mathbf{b} \end{array} \right\}.$$

The above lemma and the definition of  $\Xi_k$  directly implies the following.

**COROLLARY 2.2.** *A vector  $\mathbf{x} \in \{0, 1\}^N$  is winning if and only if there exists a unique set  $\Xi_k$  ( $k \in \{1, 2, \dots, m\}$ ) including a left truncated subvector of  $\mathbf{b}(\mathbf{x})$ .*

By a brute force method, we obtained that

$$\begin{aligned} \Xi_1 &= \{(1)\}, \\ \Xi_2 &= \{(1, 0), (0, 2)\}, \\ \Xi_3 &= \{(1, 0, 0), (0, 2, 0), (0, 1, 2), (0, 0, 3)\}, \\ \Xi_4 &= \left\{ \begin{array}{l} (1, 0, 0, 0), (0, 2, 0, 0), (0, 1, 2, 0), (0, 1, 1, 2), (0, 1, 0, 3), \\ (0, 0, 3, 0), (0, 0, 2, 2), (0, 0, 1, 3), (0, 0, 0, 4) \end{array} \right\}, \\ \Xi_5 &= \left\{ \begin{array}{l} (1, 0, 0, 0, 0)(0, 2, 0, 0, 0)(0, 1, 2, 0, 0)(0, 1, 1, 2, 0)(0, 1, 1, 1, 2) \\ (0, 1, 1, 0, 3)(0, 1, 0, 3, 0)(0, 1, 0, 2, 2)(0, 1, 0, 1, 3)(0, 1, 0, 0, 4) \\ (0, 0, 3, 0, 0)(0, 0, 2, 2, 0)(0, 0, 2, 1, 2)(0, 0, 2, 0, 3)(0, 0, 1, 3, 0) \\ (0, 0, 1, 2, 2)(0, 0, 1, 1, 3)(0, 0, 1, 0, 4)(0, 0, 0, 4, 0)(0, 0, 0, 3, 2) \\ (0, 0, 0, 2, 3)(0, 0, 0, 1, 4)(0, 0, 0, 0, 5) \end{array} \right\}. \end{aligned}$$

The following table of size of  $\Xi_k$  is obtained by a naive computer program for enumeration.

$k$	1	2	3	4	5	6	7	8	9	10	11	...
$ \Xi_k $	1	2	4	9	23	65	197	626	2056	6918	23714	...

Given an index subset  $B \subseteq \{1, 2, \dots, N\}$  and a positive integer  $b$ , we define that

$$f^b(B) = \begin{cases} \sum_{B' \subseteq B, |B'|=b} \left( \prod_{i \in B'} r_i \right) & (|B| \geq b), \\ 0 & (|B| < b). \end{cases}$$

We also define  $f^0(B) = 1$ .

Corollary 2.2 directly implies the following theorem, which gives the probability of win of a threshold strategy for odds problem with  $m$ -stoppings.

**THEOREM 2.3.** *Given a threshold strategy  $\text{Threshold}(i^{(m)}, \dots, i^{(2)}, i^{(1)})$  for odds problem with  $m$ -stoppings defined on a sequence of 0/1 random variables  $X_1, X_2, \dots, X_N$ , the corresponding probability of win is equal to*

$$\sum_{k=1}^m \left( \left( \prod_{i \in B_k \cup \dots \cup B_2 \cup B_1} q_i \right) \sum_{(b_k, \dots, b_1) \in \Xi_k} \left( f^{b_k}(B_k) f^{b_{k-1}}(B_{k-1}) \dots f^{b_1}(B_1) \right) \right).$$

**PROOF.** The definition of function  $f^b(B)$  directly implies that for any non-negative vector  $(b_k, b_{k-1}, \dots, b_1)$ ,

$$\begin{aligned} & \Pr \left[ \sum_{i \in B_{k'}} X_i = b_{k'} \quad (\forall k' \in \{k, k-1, \dots, 1\}) \right] \\ &= \left( \left( \prod_{i \in B_k} q_i \right) f^{b_k}(B_k) \right) \dots \left( \left( \prod_{i \in B_1} q_i \right) f^{b_1}(B_1) \right) \\ &= \left( \prod_{i \in B_k \cup \dots \cup B_2 \cup B_1} q_i \right) \left( f^{b_k}(B_k) f^{b_{k-1}}(B_{k-1}) \dots f^{b_1}(B_1) \right). \end{aligned}$$

From the uniqueness appearing in Corollary 2.2, the probability of win of a threshold strategy  $\text{Threshold}(i^{(m)}, \dots, i^{(2)}, i^{(1)})$  is equal to

$$\begin{aligned} & \sum_{k=1}^m \sum_{(b_k, \dots, b_1) \in \Xi_k} \Pr \left[ \sum_{i \in B_{k'}} X_i = b_{k'} \quad (\forall k' \in \{k, k-1, \dots, 1\}) \right] \\ &= \sum_{k=1}^m \left( \left( \prod_{i \in B_k \cup \dots \cup B_2 \cup B_1} q_i \right) \sum_{(b_k, \dots, b_1) \in \Xi_k} \left( f^{b_k}(B_k) f^{b_{k-1}}(B_{k-1}) \dots f^{b_1}(B_1) \right) \right). \end{aligned}$$

□

For example, when  $m = 3$ , the set  $\Xi_3$  of winning patterns includes four vectors  $\Xi_3 = \{(1, 0, 0), (0, 2, 0), (0, 1, 2), (0, 0, 3)\}$  and thus the probability of win is equal to

$$\begin{aligned} & \left( \prod_{i \in B_1} q_i \right) \left( \sum_{i \in B_1} r_i \right) + \left( \prod_{i \in B_2 \cup B_1} q_i \right) \left( \sum_{i \in B_2} r_i + \sum_{\{i_1, i_2\} \subseteq B_1} r_{i_1} r_{i_2} \right) \\ & + \left( \prod_{i \in B_3 \cup B_2 \cup B_1} q_i \right) \left( \sum_{i \in B_3} r_i + \sum_{\{i_1, i_2\} \subseteq B_2} r_{i_1} r_{i_2} \right. \\ & \quad \left. + \left( \sum_{i \in B_2} r_i \right) \left( \sum_{\{i_1, i_2\} \subseteq B_1} r_{i_1} r_{i_2} \right) + \sum_{\{i_1, i_2, i_3\} \subseteq B_1} r_{i_1} r_{i_2} r_{i_3} \right). \end{aligned}$$



**3. Mimic of Threshold Strategy.** In this section, we introduce an artificial sequence of random variables, which gives a lower bound of the probability of win of an optimal strategy for odds problem with  $m$ -stoppings defined on  $X_1, X_2, \dots, X_N$ . We assume the following property.

ASSUMPTION 1. For any  $i \in \{1, 2, \dots, N\}$ , the probability of failure  $q_i$  of  $X_i$  satisfies that  $0 < q_i < 1$  and  $\ln q_i$  is a rational number.

We can deal with a case that  $\ln q_i$  is irrational (for an index  $i$ ) by employing a sequence of rational numbers whose limiting value is equal to  $\ln q_i$ .

Assumption 1 implies that there exists a large positive integer  $\eta_*$  satisfying that  $\forall i \in \{1, 2, \dots, N\}, \exists \ell_i \in \mathbf{Z}_+, -\ln q_i = \ell_i / \eta_*$ . In the rest of this paper, we set  $\eta_*$  to the minimum positive integer satisfying the above condition. We denote  $e^{-1/\eta_*}$  by  $q_*$  for simplicity, i.e., we have

$$\forall i \in \{1, 2, \dots, N\}, \exists \ell_i \in \mathbf{Z}_+, q_i = q_*^{\ell_i}.$$

Obviously, inequalities  $0 < q_* < 1$  hold.

Now we introduce a sequence of 0/1 random variables  $Y_1, Y_2, \dots, Y_L$  satisfying  $L = \ell_1 + \ell_2 + \dots + \ell_N$  and  $\forall j \in \{1, 2, \dots, L\}, \Pr[Y_j = 0] = q_*$  (see Figure 1(a)). In this section, we show that any threshold strategy of  $Y_1, Y_2, \dots, Y_L$  for odds problem with  $m$ -stoppings satisfies that the corresponding probability of win is less than or equal to that of an optimal strategy of  $X_1, X_2, \dots, X_N$  for odds problem with  $m$ -stoppings.

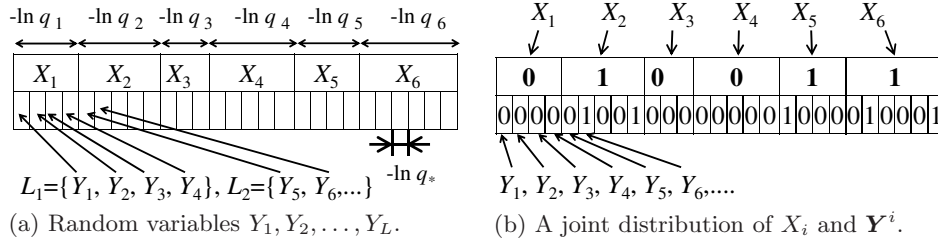


FIG 1. Mimic of threshold strategy.

Given a sequence of indices  $\mathbf{j} = (j^{(m)}, j^{(m-1)}, \dots, j^{(1)})$  satisfying inequalities  $1 \leq j^{(m)} \leq j^{(m-1)} \leq \dots \leq j^{(1)} \leq L$ , we also define a threshold strategy **Threshold**( $\mathbf{j}$ ) of  $Y_1, Y_2, \dots, Y_L$  for odds problem with  $m$ -stoppings in a similar way with a threshold strategy defined in Section 2. In the rest of this section, we discuss the probability of win of **Threshold**( $\mathbf{j}$ ).

Given a threshold strategy **Threshold**( $\mathbf{j}$ ) and an integer  $\ell \in \{1, 2, \dots, L\}$ , we define a function  $\phi_\ell : \{0, 1\}^\ell \rightarrow \{\text{win}, \text{lose}\}$  as follows. For any  $\ell$ -dimensional 0-1 vector  $\mathbf{y} \in \{0, 1\}^\ell$ , a function value  $\phi_\ell(\mathbf{y}) \in \{\text{win}, \text{lose}\}$

denotes the value of **result** obtained by executing **Threshold** ( $j$ ) by using  $\mathbf{y}$  until  $\ell$ -th iteration. It means that  $\phi_\ell(\mathbf{y})$  is equal to **win** if and only if procedure **Threshold**( $j$ ) selects last success at the end of  $\ell$ -th iteration under the assumption that  $\mathbf{y} \in \{0, 1\}^\ell$  is a vector of realized values of  $Y_1, Y_2, \dots, Y_\ell$ . More precisely, the following deterministic procedure defines a function  $\phi_\ell : \{0, 1\}^\ell \rightarrow \{\text{win}, \text{lose}\}$ .

**function**  $\phi_\ell(\mathbf{y})$

**Step 0:** Set  $j := 0$ ; **slack** := 0; **result** := **lose**.

**Step 1:** Set  $j := j + 1$ ; **slack** := **slack** +  $|\{k \in \{1, 2, \dots, m\} \mid j = j^{(k)}\}|$ .

**Step 2:** If  $[y_j = 0]$ , then goto Step 3.

Else if  $[y_j = 1 \text{ and } \text{slack} > 0]$ ,

then set **slack** := **slack** - 1; **result** := **win**.

Else (both  $y_j = 1$  and **slack** = 0 hold),

then set **result** := **lose**.

**Step 3:** If  $[j < \ell]$ , then goto Step 1.

Else, output the value of **result** and stop.

In the following, we construct a specified strategy for odds problem with  $m$ -stoppings defined on  $X_1, X_2, \dots, X_N$  which mimics a threshold strategy **Threshold**( $j$ ) defined on  $Y_1, Y_2, \dots, Y_L$ . A partition  $\{L_1, L_2, \dots, L_N\}$  (see Figure 1(a)) of index set  $\{1, 2, \dots, L\}$  is defined by

$$L_i = \begin{cases} \{1, 2, \dots, \ell_1\} & (i = 1), \\ \left\{ i' \in \{1, 2, \dots, L\} \left| \sum_{i''=1}^{i-1} \ell_{i''} < i' \leq \sum_{i''=1}^i \ell_{i''} \right. \right\} & (2 \leq i \leq N). \end{cases}$$

Assumption 1 implies that  $\forall i \in \{1, 2, \dots, N\}$ ,  $|L_i| = \ell_i > 0$  and  $q_*^{\ell_i} = q_i$ . We introduce a joint distribution of  $X_i$  and random variables  $\{Y_j \mid j \in L_i\}$  for each  $i \in \{1, 2, \dots, N\}$  satisfying that

(m1)  $\Pr[X_i = 0 \wedge Y_j = 0 \ (\forall j \in L_i)] = q_i = q_*^{\ell_i}$  and

$\Pr[X_i = 1 \wedge Y_j = 1 \ (\exists j \in L_i)] = 1 - q_i = 1 - q_*^{\ell_i}$ ,

(m2) a marginal distribution of  $X_i$  satisfies  $\Pr[X_i = 0] = q_i$ ,

(m3) set of random variables  $\{Y_j \mid j \in L_i\}$  is independent, identically distributed, and corresponding (marginal) distributions satisfy that  $\Pr[Y_j = 0] = q_* \ (\forall j \in L_i)$ .

The above condition implies that  $X_i = 0$  if and only if  $Y_j = 0$ , for all  $j \in L_i$  (see Figure 1(b) for example). A mimic strategy selects a random variable  $X_i$  if and only if  $[X_i = 1 \text{ and the threshold strategy } \text{Threshold}(j) \text{ selects the last success in } \{Y_j \mid j \in L_i\} \text{ at the end of } (\ell_1 + \ell_2 + \dots + \ell_i)\text{-th iteration}]$ . A precise definition of our mimic strategy is as follows.

Mimic ( $j$ )

**Step 0:** Set  $i := 0$ ; **result**:=lose; **last** :=  $\emptyset$ ;  $\ell := 0$ ;  $\mathbf{y} := ()$ .

**Step 1:** Set  $i := i + 1$ ;  $\ell := \ell + \ell_i$ .

Set  $x_i := \begin{cases} 0 & \text{(with probability } q_i), \\ 1 & \text{(with probability } 1 - q_i). \end{cases}$

Construct a 0-1 vector  $\mathbf{y}^i \in \{0, 1\}^{\ell_i}$  as follows.

If  $[x_i = 0]$ , we set  $\mathbf{y}^i = \mathbf{0}$ .

Else ( $x_i = 1$ ), we choose a vector  $\mathbf{y}^i \in \{0, 1\}^{\ell_i} \setminus \{\mathbf{0}\}$

with probability  $q_*^{\ell_i - y}(1 - q_*)^y / (1 - q_i)$  where  $y = \sum_{j \in L_i} y_j^i$ .

**Step 2:** Set  $\mathbf{y} := (\mathbf{y}, \mathbf{y}^i)$ ; a vector obtained by concatenating  $\mathbf{y}$  and  $\mathbf{y}^i$ .

If  $[x_i = 0]$ , then goto Step 3.

Else if  $[x_i = 1 \text{ and } \phi_\ell(\mathbf{y}) = \text{win}]$ ,

then set **last** :=  $\{i\}$ ; **result**:=win.

Else (both  $x_i = 1$  and  $\phi_\ell(\mathbf{y}) = \text{lose}$  hold),

then set **last** :=  $\emptyset$ ; **result**:=lose.

**Step 3:** If  $[i < N]$ , then goto Step 1.

Else if  $[i = N \text{ and } \text{result}=\text{win}]$ , then output the index in **last** and stop.

Else output “lose” and stop.

Since Mimic( $j$ ) is a stochastic procedure, we can deal with variables  $x_i$  and  $\mathbf{y}^i$  defined at Step 1 as realizations of random variables. Let  $\tilde{X}_i$  be a random variable corresponding to a (realized) value  $x_i$ , and  $\tilde{\mathbf{Y}}^i$  be a vector of random variables corresponding to a (realized) vector  $\mathbf{y}^i$ . It is obvious that  $\Pr[\tilde{X}_i = 0 \wedge \tilde{\mathbf{Y}}^i = \mathbf{0}] = q_i = q_*^{\ell_i}$  and  $\Pr[\tilde{X}_i = 1 \wedge \tilde{\mathbf{Y}}^i \neq \mathbf{0}] = 1 - q_i = 1 - q_*^{\ell_i}$ . Clearly, random variable  $\tilde{X}_i$  satisfies  $\Pr[\tilde{X}_i = 0] = q_i$  and a vector of random variables  $\tilde{\mathbf{Y}}^i$  satisfies

$$\Pr[\tilde{\mathbf{Y}}^i = \mathbf{0}] = \Pr[\tilde{X}_i = 0] = q_i = q_*^{\ell_i}.$$

For each vector  $\mathbf{y} \in \{0, 1\}^{\ell_i} \setminus \{\mathbf{0}\}$ , a vector of random variables  $\tilde{\mathbf{Y}}^i$  satisfies

$$\begin{aligned} \Pr[\tilde{\mathbf{Y}}^i = \mathbf{y}] &= \Pr[\tilde{\mathbf{Y}}^i = \mathbf{y} \mid \tilde{X}_i = 1] \Pr[\tilde{X}_i = 1] \\ &= \frac{q_*^{\ell_i - y}(1 - q_*)^y}{1 - q_i} (1 - q_i) = q_*^{\ell_i - y}(1 - q_*)^y \end{aligned}$$

where  $y = \sum_{j \in L_i} y_j^i$ . From the above, it is easy to show that a set of random variables in  $\tilde{\mathbf{Y}}^i$  is independent, identically distributed, and each random variable  $\tilde{Y}_j$  in vector  $\tilde{\mathbf{Y}}^i$  satisfies  $\Pr[\tilde{Y}_j = 0] = q_*$  ( $\forall j \in L_i$ ). Procedure

$\text{Mimic}(\mathbf{j})$  selects at most  $m$  indices in  $\{1, 2, \dots, N\}$ , and thus it is a strategy for odds problem with  $m$ -stoppings defined on  $X_1, X_2, \dots, X_N$ . If procedure  $\text{Mimic}(\mathbf{j})$  outputs an index, the obtained index corresponds to the last success in the vector of realized values  $(x_1, x_2, \dots, x_N)$ . A probability that  $\text{Mimic}(\mathbf{j})$  outputs an index is equal to a probability that vector of realized values  $\mathbf{y}$  satisfies  $\phi_L(\mathbf{y}) = \text{win}$ , which is the probability of win of threshold strategy  $\text{Threshold}(\mathbf{j})$ . Consequently, the win probability of  $\text{Mimic}(\mathbf{j})$  is equal to that of  $\text{Threshold}(\mathbf{j})$ . We have shown that the probability of win of an optimal strategy for odds problem with  $m$ -stoppings defined on  $X_1, X_2, \dots, X_N$  is greater than or equal to that of any threshold strategy  $\text{Threshold}(\mathbf{j})$  defined on  $Y_1, Y_2, \dots, Y_L$ . From the above discussion, we have the following.

**THEOREM 3.1.** *For any threshold strategy for odds problem with  $m$ -stoppings defined on  $Y_1, Y_2, \dots, Y_L$ , the corresponding probability of win gives a lower bound of the probability of win of an optimal strategy for odds problem with  $m$ -stoppings defined on  $X_1, X_2, \dots, X_N$ .*

We introduce a positive integer  $d$  and define a sequence of 0/1 random variables  $Y_1^d, Y_2^d, \dots, Y_L^d$  satisfying  $\Pr[Y_j = 0] = q_*^{1/d}$ . Then, we can show the following in a similar way with a discussion described above.

**COROLLARY 3.2.** *For any positive integer  $d$ , the probability of win of any threshold strategy for odds problem with  $m$ -stoppings on  $Y_1^d, Y_2^d, \dots, Y_L^d$  is less than or equal to that of an optimal strategy for odds problem with  $m$ -stoppings defined on  $X_1, X_2, \dots, X_N$ .*

Lastly, we show a lemma required in the next section.

**LEMMA 3.3.** *If a sequence of 0/1 random variables  $X_1, X_2, \dots, X_N$  satisfies  $q_1 q_2 \cdots q_N < e^{-\lambda}$ , then there exists a positive integer  $d'$  satisfying that for any integer  $d > d'$ , the total sum of a common odds  $r_*^{(d)} = (1 - q_*^{1/d})/q_*^{1/d}$  of the sequence of random variables  $Y_1^d, Y_2^d, \dots, Y_L^d$  satisfies  $L r_*^{(d)} > \lambda$ .*

**PROOF.** Let  $\lambda'$  be a value satisfying  $q_1 q_2 \cdots q_N = e^{-\lambda'}$ . From the definition of sequence  $Y_1^d, Y_2^d, \dots, Y_L^d$ , it is easy to see that  $\Pr[Y_j^d = 0] = q_*^{1/d}$  ( $\forall j$ ) and  $q_*^{L/d} = q_1 q_2 \cdots q_N = e^{-\lambda'}$ . By employing L'Hospital's rule, the total sum

of odds, denoted by  $Lr_*^{(d)}$ , satisfies

$$\begin{aligned} \lim_{d \rightarrow \infty} Lr_*^{(d)} &= \lim_{d \rightarrow \infty} L \frac{1 - q_*^{1/d}}{q_*^{1/d}} = \lim_{d \rightarrow \infty} \left( \left( \frac{-\lambda' d}{\ln q_*} \right) \left( \frac{1 - q_*^{1/d}}{q_*^{1/d}} \right) \right) \\ &\geq \frac{\lambda'}{-\ln q_*} \lim_{d \rightarrow \infty} d(1 - q_*^{1/d}) = \frac{\lambda'}{-\ln q_*} \lim_{d \rightarrow \infty} \frac{1 - q_*^{1/d}}{1/d} \\ &= \frac{\lambda'}{-\ln q_*} \lim_{d \rightarrow \infty} \frac{(\ln q_*) q_*^{1/d} d^{-2}}{-d^{-2}} = \lambda' \lim_{d \rightarrow \infty} q_*^{1/d} = \lambda' > \lambda. \end{aligned}$$

The above inequality directly implies the desired result.  $\square$

**4. Lower Bounds.** In this section, we discuss the probability of win of a threshold strategy for odds problem with  $m$ -stoppings defined on a sequence  $Y_1^d, Y_2^d, \dots, Y_L^d$  introduced at the last of the previous section.

Let  $(\lambda_1, \lambda_2, \dots, \lambda_m)$  be a solution vector of an equality system;

$$(4.1) \quad \sum_{(b_k, \dots, b_1) \in \Xi_k} \left( \frac{\lambda_k^{b_k}}{b_k!} \frac{\lambda_{k-1}^{b_{k-1}}}{b_{k-1}!} \dots \frac{\lambda_1^{b_1}}{b_1!} \right) = 1 \quad (k \in \{1, 2, \dots, m\}).$$

For example,  $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$  is a solution of the following equality system

$$\begin{aligned} \lambda_1 &= 1, \\ \lambda_2 + \frac{\lambda_1^2}{2!} &= 1, \\ \lambda_3 + \frac{\lambda_2^2}{2!} + \frac{\lambda_1^3}{3!} + \lambda_2 \frac{\lambda_1^2}{2!} &= 1, \\ \lambda_4 + \frac{\lambda_3^2}{2!} + \frac{\lambda_3 \lambda_2^2}{1! 2!} + \frac{\lambda_3 \lambda_2 \lambda_1^2}{1! 1! 2!} + \frac{\lambda_3 \lambda_1^3}{1! 3!} + \frac{\lambda_2^3}{3!} + \frac{\lambda_2^2 \lambda_1^2}{2! 2!} + \frac{\lambda_2 \lambda_1^3}{1! 3!} + \frac{\lambda_1^4}{4!} &= 1. \end{aligned}$$

By solving the above system, we obtain a solution vector  $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (1, 1/2, 11/24, 505/1152)$ . Now we show the uniqueness of a solution. The following property is discussed by Gilbert and Mosteller [13] in a setting of secretary problem.

**LEMMA 4.1.** *For any  $k \in \{1, 2, \dots, m\}$ ,  $\Xi_k$  includes unit  $k$ -vector  $\mathbf{e} = (1, 0, 0, \dots, 0)$  and every vector  $(b_k, b_{k-1}, \dots, b_1) \in \Xi_k \setminus \{\mathbf{e}\}$  satisfies  $b_k = 0$ .*

**PROOF.** From the definition of  $\Xi_k$ , it is obvious that  $\mathbf{e} \in \Xi_k$ . If a vector  $(b_k, b_{k-1}, \dots, b_1) \in \Xi_k \setminus \{\mathbf{e}\}$  satisfies  $b_k \neq 0$ , then  $b_k = 1$  and thus a left truncated subvector  $(b_{k-1}, b_{k-2}, \dots, b_1)$  is not a zero-vector and thus contained in  $\hat{\Xi}_{k-1}$ . Contradiction.  $\square$

TABLE 1

**THEOREM 4.3.** *Let  $(\lambda_1, \lambda_2, \dots, \lambda_m)$  be a unique solution of equality system (4.1). If a sequence of 0/1 random variables  $X_1, X_2, \dots, X_N$  satisfies  $\prod_{i=1}^N q_i < e^{-\sum_{k=1}^m \lambda_k}$ , then the probability of win of an optimal strategy for*

odds problem with  $m$ -stoppings defined on  $X_1, X_2, \dots, X_N$  is greater than or equal to

$$\sum_{k=1}^m e^{-\sum_{k'=1}^k \lambda_{k'}}.$$

PROOF. For any positive integer  $d$ , we introduce a sequence of 0/1 random variables  $Y_1^d, Y_2^d, \dots, Y_L^d$  (defined at the last of previous section) satisfying that a probability of failure  $\Pr[Y_j^d = 0] = q_*^{1/d}$  for each  $j \in \{1, 2, \dots, L\}$ . We denote a common odds  $(1 - q_*^{1/d})/q_*^{1/d}$  by  $r_*^{(d)}$ .

We introduce a specified threshold strategy  $\text{Threshold}(j_d^{(m)}, j_d^{(m-1)}, \dots, j_d^{(1)})$  for odds problem with  $m$ -stoppings on  $Y_1^d, Y_2^d, \dots, Y_L^d$  defined by

$$(4.2) \quad j_d^{(k)} = \min\{j \in \{1, 2, \dots, L\} \mid \lambda_1 + \lambda_2 + \dots + \lambda_k > (L - j)r_*^{(d)}\},$$

for each  $k \in \{1, 2, \dots, m\}$ . Theorem 4.2 implies inequalities  $1 \leq j_d^{(m)} \leq j_d^{(m-1)} \leq \dots \leq j_d^{(1)} \leq L$  and thus we can define a corresponding threshold strategy. Let  $\{B_{m+1}(d), B_m(d), \dots, B_1(d)\}$  be a partition of index set  $\{1, 2, \dots, L\}$  defined by

$$B_k(d) = \begin{cases} \{j \in \{1, 2, \dots, L\} \mid j_d^{(1)} \leq j \leq L\} & (k = 1), \\ \{j \in \{1, 2, \dots, L\} \mid j_d^{(k)} \leq j < j_d^{(k-1)}\} & (1 < k \leq m), \\ \{j \in \{1, 2, \dots, L\} \mid 1 \leq j < j_d^{(m)}\} & (k = m + 1). \end{cases}$$

First, we show that for any  $k \in \{1, 2, \dots, m\}$ ,

$$(4.3) \quad \lim_{d \rightarrow \infty} r_*^{(d)} |B_k(d)| = \lambda_k.$$

From the assumption  $\prod_{i=1}^N q_i < e^{-\sum_{k=1}^m \lambda_k}$ , Lemma 3.3 implies that there exists a positive integer  $d'$  satisfying that for any integer  $d > d'$ , the length  $L$  satisfies  $L > \sum_{k=1}^m \lambda_k / r_*^{(d)}$ . When  $d$  is a sufficiently large positive integer, the size of block  $B_k(d)$  satisfies

$$\frac{\lambda_k}{r_*^{(d)}} + 2 \geq |B_k(d)| \geq \frac{\lambda_k}{r_*^{(d)}} - 1 \quad (\forall k \in \{1, 2, \dots, m\})$$

and

$$\begin{aligned} \lim_{d \rightarrow \infty} (\lambda_k + 2r_*^{(d)}) &\geq \lim_{d \rightarrow \infty} r_*^{(d)} |B_k(d)| \geq \lim_{d \rightarrow \infty} (\lambda_k - r_*^{(d)}) \\ \lambda_k &\geq \lim_{d \rightarrow \infty} r_*^{(d)} |B_k(d)| \geq \lambda_k, \end{aligned}$$

since the common odds  $r_*^{(d)} = (1 - q_*^{1/d})/q_*^{1/d} > 0$  satisfies  $\lim_{d \rightarrow \infty} r_*^{(d)} = 0$ . From the above, we obtain (4.3).

Next, we show that

(4.4)

$$\lim_{d \rightarrow \infty} \prod_{j \in B_k(d)} q_*^{1/d} = e^{-\lambda_k} \quad \text{and} \quad \forall b \in \{0, 1, \dots, m\}, \lim_{d \rightarrow \infty} f^b(B_k(d)) = \frac{\lambda_k^b}{b!},$$

for each  $k \in \{1, 2, \dots, m\}$ . It is easy to see that

$$\begin{aligned} \lim_{d \rightarrow \infty} \prod_{j \in B_k(d)} q_*^{1/d} &= \lim_{d \rightarrow \infty} q_*^{|B_k(d)|/d} = \lim_{d \rightarrow \infty} \left( \frac{1}{1 + r_*^{(d)}} \right)^{|B_k(d)|} \\ &= \lim_{r_*^{(d)} \rightarrow +0} \left( \left( \frac{1}{1 + r_*^{(d)}} \right)^{\frac{1}{r_*^{(d)}}} \right)^{r_*^{(d)} |B_k(d)|} = e^{-\lambda_k}. \end{aligned}$$

When  $b = 0$ , the definition of  $f^0(B_k(d))$  says that  $f^0(B_k(d)) = 1 = \frac{\lambda_k^0}{0!}$  holds permanently. Theorem 4.2 and (4.3) say that a limiting value of  $\lim_{d \rightarrow \infty} r_*^{(d)} |B_k(d)|$  is a positive constant  $\lambda_k$ . Since the common odds satisfies  $\lim_{d \rightarrow \infty} r_*^{(d)} = 0$ , it is clear that  $\lim_{d \rightarrow \infty} |B_k(d)| = +\infty$ . When  $d$  is a sufficiently large integer, the size of block  $B_k(d)$  is greater than  $m$  and the definition of  $f^b(B_k(d))$  implies

$$\forall b \in \{1, 2, \dots, m\}, \quad \lim_{d \rightarrow \infty} f^b(B_k(d)) = \lim_{d \rightarrow \infty} \binom{|B_k(d)|}{b} (r_*^{(d)})^b.$$

It is easy to see that

$$\begin{aligned} \lim_{d \rightarrow \infty} \frac{|B_k(d)|^b}{b!} (r_*^{(d)})^b &\geq \lim_{d \rightarrow \infty} \binom{|B_k(d)|}{b} (r_*^{(d)})^b \geq \lim_{d \rightarrow \infty} \frac{(|B_k(d)| - b)^b}{b!} (r_*^{(d)})^b \\ \lim_{d \rightarrow \infty} \frac{(r_*^{(d)} |B_k(d)|)^b}{b!} &\geq \lim_{d \rightarrow \infty} \binom{|B_k(d)|}{b} (r_*^{(d)})^b \geq \lim_{d \rightarrow \infty} \frac{(r_*^{(d)} |B_k(d)| - r_*^{(d)} b)^b}{b!} \\ \frac{\lambda_k^b}{b!} &\geq \lim_{d \rightarrow \infty} \binom{|B_k(d)|}{b} (r_*^{(d)})^b \geq \frac{\lambda_k^b}{b!}. \end{aligned}$$

From the above, we obtain that  $\lim_{d \rightarrow \infty} f^b(B_k(d)) = \frac{\lambda_k^b}{b!}$ .

Now, we show our lower bound. As shown in Corollary 3.2, for any positive integer  $d$ , the probability of win of any threshold strategy for odds problem with  $m$ -stoppings defined on  $Y_1^d, Y_2^d, \dots, Y_L^d$  gives a lower bound of that



defined on  $X_1, X_2, \dots, X_N$ . The above discussions directly imply that the win probability of  $\text{Threshold}(j_d^{(m)}, j_d^{(m-1)}, \dots, j_d^{(1)})$  gives a lower bound

$$\begin{aligned} & \lim_{d \rightarrow \infty} \sum_{k=1}^m \left( \left( \prod_{i \in B_k \cup \dots \cup B_2 \cup B_1} q_i \right) \sum_{(b_k, \dots, b_1) \in \Xi_k} \left( f^{b_k}(B_k(d)) \dots f^{b_1}(B_1(d)) \right) \right) \\ &= \sum_{k=1}^m \left( \left( \prod_{k'=1}^k e^{-\lambda_{k'}} \right) \left( \sum_{(b_k, \dots, b_1) \in \Xi_k} \frac{\lambda_k^{b_k}}{b_k!} \frac{\lambda_{k-1}^{b_{k-1}}}{b_{k-1}!} \dots \frac{\lambda_1^{b_1}}{b_1!} \right) \right) = \sum_{k=1}^m e^{-\sum_{k'=1}^k \lambda_{k'}} \end{aligned}$$

where the last equality is obtained from (4.1).  $\square$

The above theorem says that; when  $m = 3, 4, 5$ , our lower bounds are  $e^{-\frac{47}{24}} + e^{-\frac{3}{2}} + e^{-1} \geq 0.7321029820$ ,  $e^{-\frac{2761}{1152}} + e^{-\frac{47}{24}} + e^{-\frac{3}{2}} + e^{-1} \geq 0.8231206726$ , and  $e^{-\frac{4162637}{1474560}} + e^{-\frac{2761}{1152}} + e^{-\frac{47}{24}} + e^{-\frac{3}{2}} + e^{-1} \geq 0.8825499145$ , respectively. Tables 2 and 3 show our lower bounds in cases  $m \leq 10$ .

In the next section, we propose an efficient method for calculating a unique solution  $(\lambda_1, \lambda_2, \dots, \lambda_m)$  of (4.1) without enumerating vectors in  $\Xi_k$  ( $k \in \{1, 2, \dots, m\}$ ).

**5. Calculating Lower Bounds Efficiently.** In this section, we describe a proof of Theorem 4.2. Our proof naturally induces an efficient technique for calculating our lower bounds.

First, we introduce a directed graph, which plays an important role in our proof of Theorem 4.2. Let  $G_m$  be a directed graph with vertex set

$$V = \{(k, k') \in \mathbf{Z}_+^2 \mid 0 \leq k \leq k' \leq m\}$$

and directed edge set

$$E = \{(v, \tilde{v}) \in V^2 \mid \exists (k, k', k''), v = (k+1, k'), \tilde{v} = (k, k''), k' \geq k''\}.$$

Figure 2 shows directed graph  $G_5$ .

For any path of length  $k$  (where  $k \in \{1, 2, \dots, m\}$ ) on  $G_m$  starting from vertex  $(k, k)$  defined by a sequence of vertices

$$((k, c_k), (k-1, c_{k-1}), \dots, (1, c_1), (0, c_0)) \quad \text{where} \quad (k, c_k) = (k, k),$$

we associate a non-negative vector  $\mathbf{b} = (b_k, b_{k-1}, \dots, b_1)$  satisfying  $b_{k'} = c_{k'} - c_{k'-1}$  ( $k' \in \{k, k-1, \dots, 1\}$ ). The above definition directly implies that  $\forall k' \in \{k, k-1, \dots, 1\}$ ,  $c_{k'-1} = k - (b_k + b_{k-1} + \dots + b_{k'})$ . Figure 3 (a) shows a path corresponding to vector  $(b_4, b_3, b_2, b_1) = (0, 2, 0, 1)$ . When a sequence of vertices  $((k, k), (k-1, c_{k-1}), \dots, (1, c_1), (0, c_0))$  forms a path on

[illegible]

TABLE 3  
Lower bounds.

$m$	$\sum_{k=1}^m e^{-\sum_{k'=1}^k \lambda_{k'}}$
1	0.3678794411
2	0.5910096013
3	0.7321029820
4	0.8231206726
5	0.8825499145
6	0.9216748810
7	0.9475883491
8	0.9648310882
9	0.9763466188
10	0.9840603638

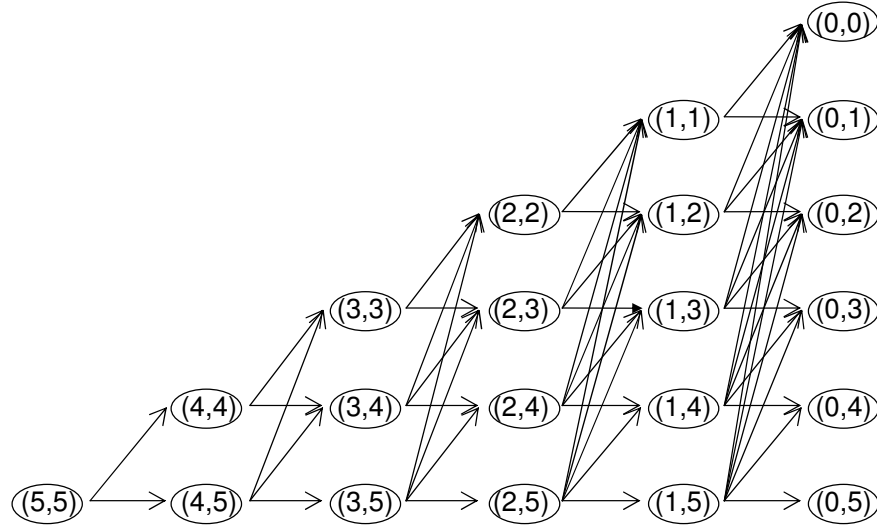


FIG 2. Directed graph  $G_5$ .

$G_m$ , then  $k' - 1 \leq c_{k'-1}$  for all  $k' \in \{k, k-1, \dots, 1\}$ , which implies that  $(b_k + b_{k-1} + \dots + b_{k'}) = k - c_{k'-1} \leq k - k' + 1$ . It is easy to see that a non-negative vector  $\mathbf{b} = (b_k, b_{k-1}, \dots, b_1)$  corresponds to a path of length  $k$  on  $G_m$  starting from vertex  $(k, k)$  if and only if  $\mathbf{b}$  satisfies

$$b_k + b_{k-1} + \dots + b_{k'} + k' \leq k + 1 \quad (k \geq \forall k' \geq 1).$$

Let us recall the definition of  $\hat{\Xi}_k$ , i.e.,

$$\hat{\Xi}_k = \left\{ (b_k, b_{k-1}, \dots, b_1) \in \mathbf{Z}_+^k \left| \begin{array}{l} \sum_{\ell=k'}^k b_\ell + k' \leq k + 1 \quad (k \geq \forall k' \geq 1), \\ \sum_{\ell=1}^k b_\ell \geq 1 \end{array} \right. \right\}.$$

The above definition implies that a non-negative vector  $\mathbf{b} = (b_k, b_{k-1}, \dots, b_1)$  corresponds to a path of length  $k$  on  $G_m$  starting from vertex  $(k, k)$  if and only if either  $\mathbf{b} \in \hat{\Xi}_k$  or  $\mathbf{b} = \mathbf{0}$ .

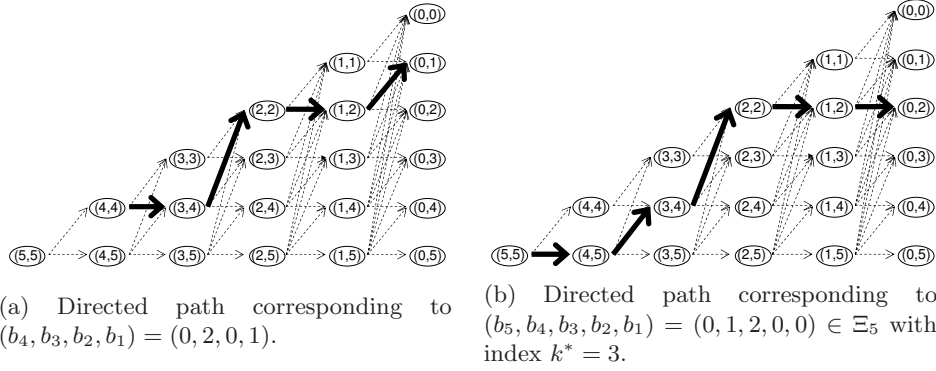


FIG 3. Directed paths corresponding to non-negative vectors.

Next lemma characterizes paths corresponding to vectors in  $\Xi_k \subseteq \hat{\Xi}_k$ .

LEMMA 5.1. *For any integer  $k \in \{1, 2, \dots, m\}$ , a non-negative vector  $\mathbf{b} = (b_k, b_{k-1}, \dots, b_1)$  is contained in  $\Xi_k$  if and only if there exists a unique index  $k^* \in \{k, k-1, \dots, 1\}$  satisfying*

- (c1)  $b_k + b_{k-1} + \dots + b_{k'} + k' < k + 1 \quad (k \geq \forall k' > k^*),$
- (c2)  $b_k + b_{k-1} + \dots + b_{k^*} + k^* = k + 1,$
- (c3)  $b_k + b_{k-1} + \dots + b_{k'} + k' < k + 1 \quad (k^* > \forall k' \geq 1),$  and
- (c4)  $b_{k^*-1} = b_{k^*-2} = \dots = b_1 = 0.$

Before proving the above lemma, we consider an example  $(b_5, b_4, b_3, b_2, b_1) = (0, 1, 2, 0, 0) \in \Xi_5$ . In this case, by setting  $k^* = 3$ , property (c2) holds, because  $b_5 + b_4 + b_3 + k^* = 0 + 1 + 2 + 3 = 6 = k + 1$ . It is easy to see that

other properties are also satisfied. Figure 3 (b) shows a corresponding path on  $G_5$ .

Property (c2) says that a path on  $G_m$  corresponding to  $\mathbf{b} \in \Xi_k$  passes vertex  $(k^* - 1, k^* - 1)$  (see Figure 4 (a)), since

$$c_{k^*-1} = k - (b_k + b_{k-1} + \dots + b_{k^*}) = k - (k + 1 - k^*) = k^* - 1.$$

Combining properties (c1), (c2) and (c3), we can say that a specified set of vertices  $\{(0, 0), (1, 1), \dots, (k, k)\}$  includes exactly two vertices  $\{(k, k), (k^* - 1, k^* - 1)\}$  in the path. From property (c4), a left truncated subvector  $(b_{k^*-1}, b_{k^*-2}, \dots, b_1)$  becomes the zero-vector, which corresponds to a sub-path from  $(k^* - 1, k^* - 1)$  to  $(0, k^* - 1)$  forming a horizontal line in Figure 4 (a).

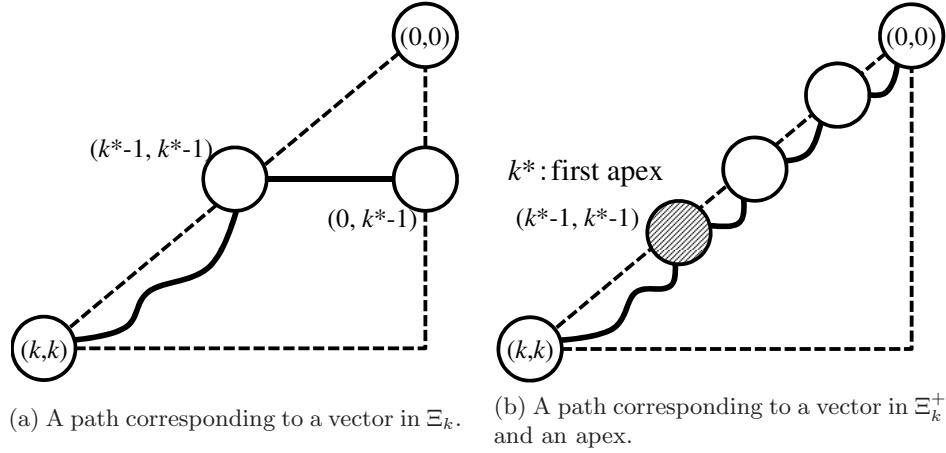


FIG 4. Directed paths on  $G_m$ .

PROOF. (Proof of Lemma 5.1.) Let  $\mathbf{b} = (b_k, b_{k-1}, \dots, b_1)$  be a vector in  $\Xi_k$ . Since  $\Xi_k \subseteq \widehat{\Xi}_k$ , vector  $\mathbf{b}$  is also an element of  $\widehat{\Xi}_k$ .

Assume on the contrary that every  $k' \in \{k, k-1, \dots, 1\}$  satisfies strict inequality  $\sum_{\ell=k'}^k b_\ell + k' < k+1$ . By setting  $k'$  to  $k$ , we obtain  $b_k + k < k+1$  and thus  $b_k = 0$ . It implies that for any  $k' \in \{k-1, k-2, \dots, 1\}$ , inequalities  $\sum_{\ell=k'}^{k-1} b_\ell + k' = \sum_{\ell=k'}^k b_\ell + k' < k+1$  hold. A left truncated subvector  $(b_{k-1}, b_{k-2}, \dots, b_1)$  of  $\mathbf{b}$  satisfies

$$\sum_{\ell=k'}^{k-1} b_\ell + k' \leq (k-1) + 1 \quad (k-1 \geq \forall k' \geq 1).$$

Since  $\mathbf{b} \in \widehat{\Xi}_k$ , we have that  $1 \leq \sum_{\ell=1}^k b_\ell = \sum_{\ell=1}^{k-1} b_\ell$ . Accordingly, a left truncated subvector  $(b_{k-1}, \dots, b_1)$  is contained in  $\widehat{\Xi}_{k-1}$ , and thus  $\mathbf{b} \notin \Xi_k$ . It is a contradiction.

From the above, there exists at least one index  $k'$  satisfying the equality  $\sum_{\ell=k'}^k b_\ell + k' = k + 1$ . We set  $k^* \in \{k, k-1, \dots, 1\}$  to the maximum (i.e., leftmost) index of  $(b_k, b_{k-1}, \dots, b_1)$  satisfying  $\sum_{\ell=k^*}^k b_\ell + k^* = k + 1$ , which directly implies properties (c1) and (c2).

Next, we show property (c4). Property (c2) implies that  $k^* - 1 \geq \forall k' \geq 1$ ,

$$\begin{aligned} \sum_{\ell=k'}^{k^*-1} b_\ell + k' &= \sum_{\ell=k'}^k b_\ell - \sum_{\ell=k^*}^k b_\ell + k' = \sum_{\ell=k'}^k b_\ell - (k + 1 - k^*) + k' \\ &\leq (k + 1) - (k + 1 - k^*) = k^* = (k^* - 1) + 1. \end{aligned}$$

If  $b_{k^*-1} + b_{k^*-2} + \dots + b_1 > 0$ , then a left truncated subvector  $(b_{k^*-1}, b_{k^*-2}, \dots, b_1)$  is contained in  $\widehat{\Xi}_{k^*-1}$ , which implies  $\mathbf{b} \notin \Xi_k$ . It is a contradiction. We have shown  $b_{k^*-1} + b_{k^*-2} + \dots + b_1 = 0$  and (c4). Property (c3) is obtained directly from (c4).

Next, we discuss the inverse implication that a non-negative vector  $\mathbf{b} = (b_k, b_{k-1}, \dots, b_1)$  satisfies (c1)-(c4). Property (c2) directly implies  $\mathbf{b} \neq \mathbf{0}$  and  $\mathbf{b} \in \widehat{\Xi}_k$ . Assume on the contrary that  $\mathbf{b} \notin \Xi_k$ . Then there exists an index  $k' \in \{k-1, k-2, \dots, 1\}$  satisfying that a left truncated subvector  $\mathbf{b}' = (b_{k'}, b_{k'-1}, \dots, b_1)$  is contained in  $\widehat{\Xi}_{k'}$ . Since  $\mathbf{b}' \neq \mathbf{0}$ , property (c4) implies  $k > k' \geq k^*$  and equivalently  $k \geq k' + 1 > k^*$ . From property (c1), an inequality

$$b_k + b_{k-1} + \dots + b_{k'+1} + (k' + 1) < k + 1$$

is obtained directly. Then, we have that

$$\begin{aligned} &b_{k'} + b_{k'-1} + \dots + b_{k^*} + k^* \\ &= (b_k + b_{k-1} + \dots + b_{k^*}) - (b_k + b_{k-1} + \dots + b_{k'+1}) + k^* \\ &> (k + 1 - k^*) - (k + 1) + (k' + 1) + k^* = k' + 1, \end{aligned}$$

which contradicts with the assumption  $\mathbf{b}' \in \widehat{\Xi}_{k'}$ .  $\square$

For any  $k \in \{1, 2, \dots, m\}$ , we introduce

$$\Xi_k^+ = \left\{ (b_k, b_{k-1}, \dots, b_1) \in \widehat{\Xi}_k \mid b_k + b_{k-1} + \dots + b_2 + b_1 = k \right\}.$$

It is clear that  $\Xi_k^+$  is a set of vectors corresponding to paths on  $G_m$  from vertex  $(k, k)$  to  $(0, 0)$ . For any vector  $\mathbf{b} \in \Xi_k^+$ , the corresponding path  $((k, k), (k-1, c_{k-1}), \dots, (1, c_1), (0, c_0))$  (where  $(0, c_0) = (0, 0)$ ) has a unique index  $k^* \in \{k, k-1, \dots, 1\}$ , called the *first apex* (see Figure 4(b)), satisfying (1)  $c_{k^*-1} = k^* - 1$  (the path passes vertex  $(k^* - 1, k^* - 1)$ ) and

(2)  $k > \forall k' \geq k^*, c_{k'} > k'$  (the path does not include any vertex in  $\{(k-1, k-1), (k-2, k-2), \dots, (k^*, k^*)\}$ ). We can partition the set of vectors in  $\Xi_k^+$  depending on their first apexes. Let  $\Xi_k^+(k^*)$  be a set of vectors in  $\Xi_k^+$  whose first apexes are  $k^*$ . For any vector  $(b_k, b_{k-1}, \dots, b_1) \in \Xi_k^+(k^*)$ , Lemma 5.1 implies that  $(b_k, b_{k-1}, \dots, b_{k^*}, 0, 0, \dots, 0) \in \mathbf{Z}_+^k$  is contained in  $\Xi_k$  and the definition of  $\Xi_k^+$  implies  $(b_{k^*-1}, b_{k^*-2}, \dots, b_1) \in \Xi_{k^*-1}^+$ .

We introduce a weight  $w(e)$  of an edge  $e = ((k+1, k'), (k, k'')) \in E$ , defined by  $w(e) = \lambda_k^{k'-k''} / (k' - k'')!$  where  $(\lambda_1, \lambda_2, \dots, \lambda_m)$  is a unique solution of (4.1). For each directed path on  $G_m$ , we define that a weight of the path is equal to the product of weights of all the edges in the path. It is easy to see that, for any vector  $(b_k, b_{k-1}, \dots, b_1) \in \Xi_k$ , the weight of the corresponding path is equal to

$$\left( \frac{\lambda_k^{b_k}}{b_k!} \frac{\lambda_{k-1}^{b_{k-1}}}{b_{k-1}!} \dots \frac{\lambda_1^{b_1}}{b_1!} \right).$$

Especially, a zero-vector induces a path forming a horizontal line in Figure 2, whose weight is equal to 1. Equality system (4.1) implies that the sum of weights of paths corresponding to vectors in  $\Xi_k$  is equal to 1.

For any vertex  $(k, k') \in V \setminus \{(0, 0)\}$ ,  $\gamma(k, k')$  denotes the sum of the weights of directed paths on  $G_m$  from vertex  $(k, k')$  to  $(0, 0)$ . We define  $\gamma(0, 0) = 1$ . The following lemma plays an important role for proving Theorem 4.2.

LEMMA 5.2. *Let  $(\lambda_1, \lambda_2, \dots, \lambda_m)$  be a unique solution of equality system (4.1). Then, for any  $k \in \{1, 2, \dots, m\}$ , the sum of weights of paths from  $(k, k)$  to  $(0, 0)$  on  $G_m$ , denoted by  $\gamma(k, k)$ , is equal to 1.*

PROOF. The definition of  $\Xi_k^+$  directly implies

$$\gamma(k, k) = \sum_{(b_k, \dots, b_1) \in \Xi_k^+} \left( \frac{\lambda_k^{b_k}}{b_k!} \frac{\lambda_{k-1}^{b_{k-1}}}{b_{k-1}!} \dots \frac{\lambda_1^{b_1}}{b_1!} \right).$$

We show  $\gamma(k, k) = 1$  by induction on  $k$ .

When  $k = 1$ ,  $\Xi_1^+ = \{(1)\}$  and  $\lambda_1 = 1$ , and thus we have a desired result.

Assume that for every  $k' \in \{1, 2, \dots, k-1\}$ , equality

$$\gamma(k', k') = \sum_{(b_{k'}, \dots, b_1) \in \Xi_{k'}^+} \left( \frac{\lambda_{k'}^{b_{k'}}}{b_{k'}!} \frac{\lambda_{k'-1}^{b_{k'-1}}}{b_{k'-1}!} \dots \frac{\lambda_1^{b_1}}{b_1!} \right) = 1$$

holds. For each  $k^* \in \{k, k-1, \dots, 1\}$ ,  $\Xi_k^+(k^*)$  denotes a set of vectors in  $\Xi_k^+$  whose first apexes are  $k^*$ . Similarly, we define that for any index  $k^* \in \{k, k-$

$1, \dots, 1\}$ ,  $\Xi_k(k^*)$  is a set of vectors in  $\Xi_k$  which have  $k^*$  as a unique index satisfying properties (c1)-(c4) in Lemma 5.1. As discussed above, every vector  $(b_k, b_{k-1}, \dots, b_1) \in \Xi_k^+(k^*)$  satisfies that  $(b_k, b_{k-1}, \dots, b_{k^*}, 0, 0, \dots, 0) \in \mathbf{Z}_+^k$  is contained in  $\Xi_k(k^*)$  and  $(b_{k^*-1}, b_{k^*-2}, \dots, b_1) \in \Xi_{k^*-1}^+$ . Conversely, for any pair of vectors  $(\mathbf{b}, \widehat{\mathbf{b}}) \in \Xi_k(k^*) \times \widehat{\Xi}_{k^*-1}$ , it is easy to see that

$$(b_k, b_{k-1}, \dots, b_{k^*}, \widehat{b}_{k^*-1}, \widehat{b}_{k^*-2}, \dots, \widehat{b}_1) \in \Xi_k^+(k^*).$$

The induction hypothesis and Lemma 5.1 implies

$$\begin{aligned} \gamma(k, k) &= \sum_{(b_k, \dots, b_1) \in \Xi_k^+} \left( \frac{\lambda_k^{b_k}}{b_k!} \frac{\lambda_{k-1}^{b_{k-1}}}{b_{k-1}!} \dots \frac{\lambda_1^{b_1}}{b_1!} \right) \\ &= \sum_{k^*=1}^k \sum_{(b_k, \dots, b_1) \in \Xi_k^+(k^*)} \left( \frac{\lambda_k^{b_k}}{b_k!} \frac{\lambda_{k-1}^{b_{k-1}}}{b_{k-1}!} \dots \frac{\lambda_1^{b_1}}{b_1!} \right) \\ &= \sum_{k^*=1}^k \left( \left( \sum_{(b_k, \dots, b_1) \in \Xi_k(k^*)} \frac{\lambda_k^{b_k}}{b_k!} \dots \frac{\lambda_{k^*}^{b_{k^*}}}{b_{k^*}!} \right) \left( \sum_{(\widehat{b}_{k^*-1}, \dots, \widehat{b}_1) \in \widehat{\Xi}_{k^*-1}^+} \frac{\widehat{\lambda}_{k^*-1}^{\widehat{b}_{k^*-1}}}{\widehat{b}_{k^*-1}!} \dots \frac{\widehat{\lambda}_1^{\widehat{b}_1}}{\widehat{b}_1!} \right) \right) \\ &= \sum_{k^*=1}^k \left( \left( \sum_{(b_k, \dots, b_1) \in \Xi_k(k^*)} \frac{\lambda_k^{b_k}}{b_k!} \frac{\lambda_{k-1}^{b_{k-1}}}{b_{k-1}!} \dots \frac{\lambda_{k^*}^{b_{k^*}}}{b_{k^*}!} \right) \gamma(k^* - 1, k^* - 1) \right) \\ &= \sum_{k^*=1}^k \sum_{(b_k, \dots, b_1) \in \Xi_k(k^*)} \left( \frac{\lambda_k^{b_k}}{b_k!} \frac{\lambda_{k-1}^{b_{k-1}}}{b_{k-1}!} \dots \frac{\lambda_{k^*}^{b_{k^*}}}{b_{k^*}!} \right) \\ &= \sum_{k^*=1}^k \sum_{(b_k, \dots, b_1) \in \Xi_k(k^*)} \left( \frac{\lambda_k^{b_k}}{b_k!} \frac{\lambda_{k-1}^{b_{k-1}}}{b_{k-1}!} \dots \frac{\lambda_{k^*}^{b_{k^*}}}{b_{k^*}!} \frac{\lambda_{k^*-1}^0}{0!} \dots \frac{\lambda_1^0}{0!} \right) \\ &= \sum_{k^*=1}^k \sum_{(b_k, \dots, b_1) \in \Xi_k(k^*)} \left( \frac{\lambda_k^{b_k}}{b_k!} \frac{\lambda_{k-1}^{b_{k-1}}}{b_{k-1}!} \dots \frac{\lambda_{k^*}^{b_{k^*}}}{b_{k^*}!} \frac{\lambda_{k^*-1}^{b_{k^*-1}}}{b_{k^*-1}!} \dots \frac{\lambda_1^{b_1}}{b_1!} \right) \\ &= \sum_{(b_k, \dots, b_1) \in \Xi_k} \left( \frac{\lambda_k^{b_k}}{b_k!} \frac{\lambda_{k-1}^{b_{k-1}}}{b_{k-1}!} \dots \frac{\lambda_1^{b_1}}{b_1!} \right) = 1, \end{aligned}$$

where we define that if  $k^* = 1$ , then the following equality

$$\left( \sum_{(\widehat{b}_{k^*-1}, \dots, \widehat{b}_1) \in \widehat{\Xi}_{k^*-1}^+} \frac{\widehat{\lambda}_{k^*-1}^{\widehat{b}_{k^*-1}}}{\widehat{b}_{k^*-1}!} \dots \frac{\widehat{\lambda}_1^{\widehat{b}_1}}{\widehat{b}_1!} \right) = \gamma(0, 0) = 1$$

holds for simplicity, and the last equality is obtained from (4.1).  $\square$



Now we show a property, which directly induces Theorem 4.2.

**THEOREM 5.3.** *Let  $(\lambda_1, \lambda_2, \dots, \lambda_m)$  be a unique solution of (4.1). For any  $k \in \{1, 2, \dots, m\}$ ,  $\lambda_k > 0$  and  $1 = \gamma(k, k) > \gamma(k, k+1) > \dots > \gamma(k, m)$ .*

**PROOF.** (Proof of Theorem 5.3 and Theorem 4.2.)

We show Theorem 5.3 by induction on  $k$ . When  $k = 1$ , it is obvious, because  $\gamma(1, k) = \lambda_1^k/k!$  and  $\lambda_1 = 1$ .

Assume that  $1 = \gamma(k-1, k-1) > \gamma(k-1, k) > \dots > \gamma(k-1, m)$ . Lemma 5.2 directly implies

$$1 = \gamma(k, k) = \frac{\lambda_k^1}{1!} \gamma(k-1, k-1) + \frac{\lambda_k^0}{0!} \gamma(k-1, k) = \lambda_k + \gamma(k-1, k).$$

The induction hypothesis induces that

$$(5.1) \quad \lambda_k = 1 - \gamma(k-1, k) = \gamma(k-1, k-1) - \gamma(k-1, k) > 0.$$

In the rest of this proof, we denote  $\lambda_k^b/b!$  by  $w(b)$  and  $\gamma(k-1, k')$  by  $\gamma'(k')$  for simplicity. From Lemma 5.2, equality  $\gamma'(k-1) = 1$  holds. For any  $k' \in \{k, k+1, \dots, m-1\}$ , the induction hypothesis implies

$$\begin{aligned} & \gamma(k, k') - \gamma(k, k'+1) \\ &= w(0)\gamma'(k') + w(1)\gamma'(k'-1) + \dots + w(k'-k+1)\gamma'(k-1) \\ & \quad - \left( w(0)\gamma'(k'+1) + w(1)\gamma'(k') + \dots + w(k'-k+1)\gamma'(k) \right) \\ &= w(0)\left(\gamma'(k') - \gamma'(k'+1)\right) + w(1)\left(\gamma'(k'-1) - \gamma'(k')\right) + \dots \\ & \quad \dots + w(k'-k)\left(\gamma'(k) - \gamma'(k+1)\right) + w(k'-k+1)\left(\gamma'(k-1) - \gamma'(k)\right) \\ & \quad - w(k'-k+2)\gamma'(k-1) \\ &> w(k'-k+1)\left(\gamma'(k-1) - \gamma'(k)\right) - w(k'-k+2)\gamma'(k-1) \\ &= w(k'-k+1)\left(1 - \gamma(k-1, k)\right) - w(k'-k+2) \\ &= w(k'-k+1)\lambda_k - w(k'-k+2) \quad (\text{obtained from (5.1)}) \\ &= w(k'-k+1)\left(\lambda_k - \frac{\lambda_k}{k'-k+2}\right) \\ &= w(k'-k+1)\lambda_k \left(1 - \frac{1}{k'-k+2}\right) > 0. \end{aligned}$$

Here we note that the strict inequalities appearing above are obtained by the positivity of  $\lambda_k$ .  $\square$

The above proof directly induces a dynamic programming technique for calculating  $(\lambda_1, \lambda_2, \dots, \lambda_m)$  efficiently. As shown in (5.1), we can calculate  $\lambda_k$  by  $1 - \gamma(k-1, k)$ . Thus, we only need to calculate  $\gamma(k, k')$  for each vertex  $(k, k') \in V$  sequentially as follows.

**Algorithm A**

**Step 0:** Set  $k := 1$ ;  $\lambda_1 := 1$ ;  $\gamma(1, k') := 1/k'!$  for all  $k' \in \{1, 2, \dots, m\}$ .

**Step 1:** Set  $k := k + 1$ ;  $\gamma(k, k) := 1$ ;  $\lambda_k := 1 - \gamma(k-1, k)$ .

For each  $k' \in \{k+1, \dots, m\}$ , set  $\gamma(k, k') := \sum_{c=k}^{k'} \frac{\lambda_k^{c-k}}{(c-k)!} \gamma(k-1, c)$ .

**Step 2:** If  $k = m$ , then stop and output  $(\lambda_1, \lambda_2, \dots, \lambda_m)$ . Else, goto Step 1.

We can calculate a weight  $\frac{\lambda_k^{c-k}}{(c-k)!}$  in Step 1 efficiently by using recurrence relation  $\frac{\lambda_k^{c-k}}{(c-k)!} = \left( \frac{\lambda_k}{c-k} \right) \frac{\lambda_k^{c-k-1}}{(c-k-1)!}$ . Then, the total number of basic arithmetic operations required in Algorithm A is bounded by  $O(m^3)$ .

At the last of this section, we discuss a relation between our lower bounds and Poisson distributions defined by  $(\lambda_1, \lambda_2, \dots, \lambda_m)$ . We introduce a Markov chain defined on a state space

$$\Omega = \{(k, k') \mid k \in \{0, 1, \dots, m\} \text{ and } k' \in \{m, m-1, \dots, 0, -1, \dots\}\}.$$

For any pair of states  $v, v' \in \Omega$ , we define a transition probability

$$p_{vv'} = \begin{cases} \frac{\lambda_k^{k'-k''}}{(k'-k'')!} & (\text{if } \exists(k, k', k''), v = (k+1, k'), v' = (k, k'') \text{ and } k' \geq k''), \\ 0 & (\text{otherwise}). \end{cases}$$

If we set  $(\lambda_1, \lambda_2, \dots, \lambda_m)$  to a unique solution of (4.1), the  $k$ -th term  $e^{-\sum_{k'=1}^k \lambda_{k'}}$  of our lower bound is equal to the probability of sample path  $((k, k), (k-1, k), (k-2, k), \dots, (0, k))$ , which corresponds to path forming a horizontal line in Figure 2. Lemma 5.2 implies that for any  $k \in \{1, 2, \dots, m\}$

$$e^{-\sum_{k'=1}^k \lambda_{k'}} \sum_{(b_k, b_{k-1}, \dots, b_1) \in \Xi_k^+} \left( \frac{\lambda_k^{b_k}}{b_k!} \frac{\lambda_{k-1}^{b_{k-1}}}{b_{k-1}!} \dots \frac{\lambda_1^{b_1}}{b_1!} \right) = e^{-\sum_{k'=1}^k \lambda_{k'}}$$

The above equality says that the  $k$ -th term of our lower bound is also equal to the total sum of probabilities of sample paths

$$((k, k), (k-1, c_{k-1}), (k-2, c_{k-2}), \dots, (1, c_1), (0, 0))$$

satisfying that  $c_{k'} \geq k'$  for all  $k' \in \{k-1, k-2, \dots, 1\}$ .

**6. Tightness of Lower Bounds.** In this section, we show the tightness of our lower bounds obtained in Theorem 4.3. For any positive  $r > 0$ , we introduce a sequence of 0/1 random variables  $X_1^r, X_2^r, \dots, X_N^r$  satisfying  $\Pr[X_i^r = 0] = q = 1/(1+r)$  ( $\forall i \in \{1, 2, \dots, N\}$ ), where  $r$  is called a *common odds*. Ano, Kakinuma, and Miyoshi [3] showed that an optimal strategy for odds problem with  $m$ -stoppings is attained by a threshold strategy. We denote an optimal threshold strategy for odds problem with  $m$ -stoppings on  $X_1^r, X_2^r, \dots, X_N^r$  by  $\text{Threshold}(i_r^{(m)}, i_r^{(m-1)}, \dots, i_r^{(1)})$ , which is dependent on common odds  $r$ . A partition  $\{B_{m+1}(r), B_m(r), \dots, B_1(r)\}$  of index set  $\{1, 2, \dots, N\}$  is defined by

$$B_k(r) = \begin{cases} \{i \in \{1, 2, \dots, N\} \mid i_r^{(1)} \leq i \leq N\} & (k = 1), \\ \{i \in \{1, 2, \dots, N\} \mid i_r^{(k)} \leq i < i_r^{(k-1)}\} & (1 < k \leq m), \\ \{i \in \{1, 2, \dots, N\} \mid 1 \leq i < i_r^{(m)}\} & (k = m+1). \end{cases}$$

We show a lemma, which is a building block of our proof of tightness.

**LEMMA 6.1.** *Let  $(\lambda_1, \lambda_2, \dots, \lambda_m)$  be a vector consisting of all positive entries. Assume that a threshold strategy  $\text{Threshold}(i_r^{(m)}, i_r^{(m-1)}, \dots, i_r^{(1)})$  and a corresponding partition  $\{B_{m+1}(r), B_m(r), \dots, B_1(r)\}$  satisfy a condition*

$$(6.1) \quad \lim_{r \rightarrow +0} r|B_k(r)| = \lambda_k,$$

for each  $k \in \{1, 2, \dots, m\}$ . Then the partition also satisfies

$$(6.2) \quad \lim_{r \rightarrow +0} q^{|B_k(r)|} = e^{-\lambda_k} \text{ and } \forall b \in \{0, 1, \dots, m\}, \lim_{r \rightarrow +0} f^b(B_k(r)) = \frac{\lambda_k^b}{b!},$$

for each  $k \in \{1, 2, \dots, m\}$ .

**PROOF.** It obvious that for any  $k \in \{1, 2, \dots, m\}$ ,

$$\lim_{r \rightarrow +0} q^{|B_k(r)|} = \lim_{r \rightarrow +0} \left( \left( \frac{1}{1+r} \right)^{1/r} \right)^{r|B_k(r)|} = e^{-\lambda_k}.$$

When  $b = 0$ , the definition of  $f^0(B_k(r))$  says that  $f^0(B_k(r)) = 1 = \frac{\lambda_k^0}{0!}$  holds permanently. Next, we consider cases  $b \in \{1, 2, 3, \dots, m\}$ . Since a limiting value  $\lim_{r \rightarrow +0} r|B_k(r)|$  is a positive constant  $\lambda_k$ , it is clear that  $\lim_{r \rightarrow +0} |B_k(r)| = +\infty$ . There exists a sufficiently small positive number  $r'$  such that  $0 < \forall r < r'$ , the size of  $B_k(r)$  exceeds  $m$ , and thus

$$\lim_{r \rightarrow +0} f^b(B_k(r)) = \lim_{r \rightarrow +0} \left( \frac{|B_k(r)|}{b} \right) r^b \quad (\forall b \in \{1, 2, \dots, m\}).$$

It is easy to see that

$$\begin{aligned} \lim_{r \rightarrow +0} \frac{(|B_k(r)| - b)^b r^b}{b!} &\leq \lim_{r \rightarrow +0} \left( \frac{|B_k(r)|}{b} \right)^b r^b \leq \lim_{r \rightarrow +0} \frac{|B_k(r)|^b r^b}{b!} \\ \frac{\lambda_k^b}{b!} &\leq \lim_{r \rightarrow +0} \left( \frac{|B_k(r)|}{b} \right)^b r^b \leq \frac{\lambda_k^b}{b!}. \end{aligned}$$

Thus, we have shown  $\lim_{r \rightarrow +0} f^b(B_k(r)) = \frac{\lambda_k^b}{b!}$ .  $\square$

Now we show the tightness of our lower bound.

**THEOREM 6.2.** *Let  $(\lambda_1, \lambda_2, \dots, \lambda_m)$  be a unique solution of (4.1). For any positive  $r > 0$ , we introduce a sequence of random variables  $X_1^r, X_2^r, \dots, X_N^r$  with a common odds  $r > 0$  satisfying  $q^N < e^{-\sum_{k=1}^m \lambda_k}$  where  $q = 1/(1+r)$ . A probability of win  $P_r^{(\text{win})}(m)$  of an optimal strategy for odds problem with  $m$ -stoppings defined on the sequence  $X_1^r, X_2^r, \dots, X_N^r$  satisfies*

$$\lim_{r \rightarrow +0} P_r^{(\text{win})}(m) = \sum_{k=1}^m e^{-\sum_{k'=1}^k \lambda_{k'}}.$$

**PROOF.** First, we discuss the length  $N$  of a given sequence. Let  $\lambda'$  be a positive number satisfying  $q^N = e^{-\lambda'}$ . By employing L'Hospital's rule, assumption  $q^N = e^{-\lambda'} < e^{-\sum_{k=1}^m \lambda_k}$  implies

(6.3)

$$\lim_{r \rightarrow +0} Nr = \lim_{r \rightarrow +0} \frac{-\lambda' r}{\ln q} = \lim_{r \rightarrow +0} \frac{\lambda' r}{\ln(1+r)} = \lim_{r \rightarrow +0} \lambda'(1+r) = \lambda' > \sum_{k'=1}^m \lambda_{k'}.$$

Ano, Kakinuma, and Miyoshi [3] showed that an optimal strategy for odds problem with  $m$ -stoppings on  $X_1^r, X_2^r, \dots, X_N^r$  is attained by an optimal threshold strategy  $\text{Threshold}(i_r^{(m)}, i_r^{(m-1)}, \dots, i_r^{(1)})$  satisfying that for any index  $k \in \{1, 2, \dots, m\}$ ,  $\text{Threshold}(i_r^{(k)}, i_r^{(k-1)}, \dots, i_r^{(1)})$  is also an optimal strategy for  $k$ -stopping problem.

In the following, we show (6.1) in Lemma 6.1 (and (6.2), simultaneously) by induction on  $k$ . When  $k = 1$ ,  $\text{Threshold}(i_r^{(1)})$  is optimal to single-stopping problem. From property (6.3), the sum total of odds  $Nr > \lambda_1 = 1$ , when  $r$  is a sufficiently small positive. Bruss' Sum the Odds Theorem implies  $1 - r \leq r|B_1(r)| \leq 1 + r$ . Accordingly, we have

$$\begin{aligned} \lim_{r \rightarrow +0} (1 - r) &\leq \lim_{r \rightarrow +0} r|B_1(r)| \leq \lim_{r \rightarrow +0} (1 + r) \\ 1 &\leq \lim_{r \rightarrow +0} r|B_1(r)| \leq 1, \end{aligned}$$

and  $\lim_{r \rightarrow +0} r|B_1(r)| = 1 = \lambda_1$ . Lemma 6.1 implies that  $\forall b \in \{0, 1, 2, \dots, m\}$ ,

$$\lim_{r \rightarrow +0} f^b(B_1(r)) = \frac{\lambda_1^b}{b!}.$$

Now we describe  $k$ -th induction step (where  $k \leq m$ ) under the assumption that for any  $k' \in \{1, 2, \dots, k-1\}$ ,

(6.4)

$$\lim_{r \rightarrow +0} r|B_{k'}(r)| = \lambda_{k'} \text{ and } \forall b \in \{0, 1, 2, \dots, m\}, \lim_{r \rightarrow +0} f^b(B_{k'}(r)) = \frac{\lambda_{k'}^b}{b!}.$$

Let  $\mathbf{e}$  be a unit  $k$ -vector  $(1, 0, 0, \dots, 0)$ . Lemma 4.1 says that  $\mathbf{e} \in \Xi_k$  and every vector  $(b_k, b_{k-1}, \dots, b_1) \in \Xi_k \setminus \{\mathbf{e}\}$  satisfies  $b_k = 0$ . Thus, we have

$$\begin{aligned} & \sum_{(b_k, \dots, b_1) \in \Xi_k} \left( f^{b_k}(B_k(r)) f^{b_{k-1}}(B_{k-1}(r)) \cdots f^{b_1}(B_1(r)) \right) \\ &= r|B_k(r)| + \sum_{(b_k, \dots, b_1) \in \Xi_k \setminus \{\mathbf{e}\}} \left( f^{b_{k-1}}(B_{k-1}(r)) f^{b_{k-2}}(B_{k-2}(r)) \cdots f^{b_1}(B_1(r)) \right). \end{aligned}$$

The induction hypothesis (6.4) and equality system (4.1) imply

$$\begin{aligned} (6.5) \quad & \lim_{r \rightarrow +0} \left( \sum_{(b_k, \dots, b_1) \in \Xi_k \setminus \{\mathbf{e}\}} \left( f^{b_{k-1}}(B_{k-1}(r)) f^{b_{k-2}}(B_{k-2}(r)) \cdots f^{b_1}(B_1(r)) \right) \right) \\ &= \sum_{(b_k, \dots, b_1) \in \Xi_k \setminus \{\mathbf{e}\}} \left( \frac{\lambda_{k-1}^{b_{k-1}} \lambda_{k-2}^{b_{k-2}} \cdots \lambda_1^{b_1}}{b_{k-1}! b_{k-2}! \cdots b_1!} \right) \\ &= \sum_{(b_k, \dots, b_1) \in \Xi_k \setminus \{\mathbf{e}\}} \left( \frac{\lambda_k^0 \lambda_{k-1}^{b_{k-1}} \lambda_{k-2}^{b_{k-2}} \cdots \lambda_1^{b_1}}{0! b_{k-1}! b_{k-2}! \cdots b_1!} \right) \\ &= \sum_{(b_k, \dots, b_1) \in \Xi_k} \left( \frac{\lambda_k^{b_k} \lambda_{k-1}^{b_{k-1}} \lambda_{k-2}^{b_{k-2}} \cdots \lambda_1^{b_1}}{b_k! b_{k-1}! b_{k-2}! \cdots b_1!} \right) - \left( \frac{\lambda_k^1 \lambda_{k-1}^0 \lambda_{k-2}^0 \cdots \lambda_1^0}{1! 0! 0! \cdots 0!} \right) \\ &= 1 - \lambda_k. \end{aligned}$$

Now we employ a one-stage look-ahead approach [1] to complete  $k$ -th induction step. For each index  $i$  satisfying  $1 \leq i \leq i_r^{(k-1)}$ , we introduce a threshold strategy  $\text{Threshold}(i, i_r^{(k-1)}, i_r^{(k-2)}, \dots, i_r^{(1)})$  for  $k$ -stopping problem. We denote a corresponding probability of win by  $P_r^{(\text{win})}(k, i)$ . A difference of win probabilities of threshold strategies  $\text{Threshold}(i-1, i_r^{(k-1)}, \dots,$

$i_r^{(1)}$ ) and  $\text{Threshold}(i, i_r^{(k-1)}, \dots, i_r^{(1)})$  satisfies

$$\begin{aligned}
& P_r^{(\text{win})}(k, i-1) - P_r^{(\text{win})}(k, i) \\
&= \left( \prod_{i'=i-1}^N q \right) \left( (i_r^{(k-1)} - i + 1)r + \sum_{(b_k, \dots, b_1) \in \Xi_k \setminus \{\mathbf{e}\}} (f^{b_{k-1}}(B_{k-1}(r)) \cdots f^{b_1}(B_1(r))) \right) \\
&\quad - \left( \prod_{i'=i}^N q \right) \left( (i_r^{(k-1)} - i)r + \sum_{(b_k, \dots, b_1) \in \Xi_k \setminus \{\mathbf{e}\}} (f^{b_{k-1}}(B_{k-1}(r)) \cdots f^{b_1}(B_1(r))) \right) \\
&= q^{N-i+1} \left( (i_r^{(k-1)} - i + 1)qr - (i_r^{(k-1)} - i)r \right. \\
&\quad \left. + (q-1) \sum_{(b_k, \dots, b_1) \in \Xi_k \setminus \{\mathbf{e}\}} (f^{b_{k-1}}(B_{k-1}(r)) \cdots f^{b_1}(B_1(r))) \right) \\
&= q^{N-i+1} \left( (i_r^{(k-1)} - i)r(q-1) + qr \right. \\
&\quad \left. + (q-1) \sum_{(b_k, \dots, b_1) \in \Xi_k \setminus \{\mathbf{e}\}} (f^{b_{k-1}}(B_{k-1}(r)) \cdots f^{b_1}(B_1(r))) \right) \\
&= q^{N-i+1}(1-q) \left( (i - i_r^{(k-1)})r + 1 \right. \\
&\quad \left. - \sum_{(b_k, \dots, b_1) \in \Xi_k \setminus \{\mathbf{e}\}} (f^{b_{k-1}}(B_{k-1}(r)) \cdots f^{b_1}(B_1(r))) \right).
\end{aligned}$$

From the above, it is easy to see that  $P_r^{(\text{win})}(k, i-1) - P_r^{(\text{win})}(k, i)$  is strictly increasing with  $i$  and thus a sequence of win probabilities

$$(P_r^{(\text{win})}(k, 1), P_r^{(\text{win})}(k, 2), \dots, P_r^{(\text{win})}(k, i_r^{(k-1)}))$$

is unimodal. Let us consider a pair  $P_r^{(\text{win})}(k, i_r^{(k-1)} - 1)$  and  $P_r^{(\text{win})}(k, i_r^{(k-1)})$ . Equality (6.5) implies

$$\begin{aligned}
& \lim_{r \rightarrow +0} \left( (i_r^{(k-1)} - i_r^{(k-1)})r + 1 \right. \\
& \quad \left. - \sum_{(b_k, \dots, b_1) \in \Xi_k \setminus \{\mathbf{e}\}} (f^{b_{k-1}}(B_{k-1}(r)) \cdots f^{b_1}(B_1(r))) \right) \\
&= \lim_{r \rightarrow +0} \left( 1 - \sum_{(b_k, \dots, b_1) \in \Xi_k \setminus \{\mathbf{e}\}} (f^{b_{k-1}}(B_{k-1}(r)) \cdots f^{b_1}(B_1(r))) \right) \\
&= 1 - (1 - \lambda_k) = \lambda_k > 0,
\end{aligned}$$

and thus there exists a positive  $r'$  satisfying

$$(6.6) \quad 0 < \forall r < r', \quad P_r^{(\text{win})}(k, i_r^{(k-1)} - 1) - P_r^{(\text{win})}(k, i_r(k)) > 0.$$

Next, we consider a pair  $P_r^{(\text{win})}(k, 2)$  and  $P_r^{(\text{win})}(k, 3)$ . When  $r$  is a sufficiently small positive, property (6.3) implies  $Nr > \sum_{k'=1}^m \lambda_{k'}$  and

$$\begin{aligned} N &= i_r^{(k-1)} - 1 + |B_{k-1}(r)| + |B_{k-2}(r)| + \cdots + |B_1(r)| \\ 3 - i_r^{(k-1)} &= 2 - N + |B_{k-1}(r)| + |B_{k-2}(r)| + \cdots + |B_1(r)| \\ \lim_{r \rightarrow +0} (3 - i_r^{(k-1)})r &= \lim_{r \rightarrow +0} \left( 2r - Nr + \sum_{k'=1}^{k-1} r|B_{k'}(r)| \right) \\ &\leq \lim_{r \rightarrow +0} \left( 2r - \sum_{k'=1}^m \lambda_{k'} + \sum_{k'=1}^{k-1} r|B_{k'}(r)| \right) \\ &= - \sum_{k'=1}^m \lambda_{k'} + \sum_{k'=1}^{k-1} \lambda_{k'} = - \sum_{k'=k}^m \lambda_{k'}, \end{aligned}$$

where the penultimate equality is obtained from the induction hypothesis (6.4). Accordingly, we have

$$\begin{aligned} \lim_{r \rightarrow +0} &\left( (3 - i_r^{(k-1)})r + 1 - \sum_{(b_k, \dots, b_1) \in \Xi_k \setminus \{e\}} (f^{b_{k-1}}(B_{k-1}(r)) \cdots f^{b_1}(B_1(r))) \right) \\ &\leq \left( - \sum_{k'=k}^m \lambda_{k'} + 1 - (1 - \lambda_k) \right) = - \sum_{k'=k}^m \lambda_{k'} + \lambda_k \leq 0, \end{aligned}$$

and thus there exists a positive  $r''$  satisfying

$$(6.7) \quad 0 < \forall r < r'', \quad P^{(\text{win})}(k, 2) - P^{(\text{win})}(k, 3) \leq 0.$$

From inequalities (6.6) and (6.7), we have that  $0 < \forall r < \min\{r', r''\}$ ,

$$P_r^{(\text{win})}(k, 1) < P_r^{(\text{win})}(k, 2) \text{ and } P_r^{(\text{win})}(k, i_r^{(k-1)} - 1) > P_r^{(\text{win})}(k, i_r^{(k-1)})$$

hold, and thus  $\exists i_r^* \in \{3, 4, \dots, i_r^{(k-1)} - 1\}$ ,

$$\begin{aligned} P^{(\text{win})}(k, 1) &< P^{(\text{win})}(k, 2) \\ &< \cdots < P_r^{(\text{win})}(k, i_r^* - 2) < P_r^{(\text{win})}(k, i_r^* - 1) \geq P_r^{(\text{win})}(k, i_r^*) \\ &> \cdots > P^{(\text{win})}(k, i_r^{(k-1)} - 1) > P^{(\text{win})}(k, i_r^{(k-1)}). \end{aligned}$$

An optimal threshold strategy is attained by either  $\text{Threshold}(i_r^*, i_r^{(k-1)}, \dots, i_r^{(1)})$  or  $\text{Threshold}(i_r^* - 1, i_r^{(k-1)}, \dots, i_r^{(1)})$  and thus  $i_r^* - 1 \leq i_r^{(k)} \leq i_r^*$ . Clearly from  $P_r^{(\text{win})}(k, i_r^* - 1) \geq P_r^{(\text{win})}(k, i_r^*)$ , index  $i_r^*$  satisfies

$$r(i_r^{(k-1)} - i_r^*) \leq 1 - \sum_{(b_k, \dots, b_1) \in \Xi_k \setminus \{e\}} (f^{b_{k-1}}(B_{k-1}(r)) \cdots f^{b_1}(B_1(r))).$$

From the above, we obtain that

$$\begin{aligned}
& \lim_{r \rightarrow +0} r|B_k(r)| \\
&= \lim_{r \rightarrow +0} r(i_r^{(k-1)} - i_r^{(k)}) \leq \lim_{r \rightarrow +0} r(i_r^{(k-1)} - (i_r^* - 1)) \\
&\leq \lim_{r \rightarrow +0} \left( 1 - \sum_{(b_k, \dots, b_1) \in \Xi_k \setminus \{\mathbf{e}\}} (f^{b_{k-1}}(B_{k-1}(r)) \cdots f^{b_1}(B_1(r))) + r \right) \\
&= 1 - (1 - \lambda_k) = \lambda_k.
\end{aligned}$$

Since  $P_r^{(\text{win})}(k, i_r^* - 2) < P_r^{(\text{win})}(k, i_r^* - 1)$ , index  $i_r^*$  satisfies

$$r(i_r^{(k-1)} - (i_r^* - 1)) > 1 - \sum_{(b_k, \dots, b_1) \in \Xi_k \setminus \{\mathbf{e}\}} (f^{b_{k-1}}(B_{k-1}(r)) \cdots f^{b_1}(B_1(r))).$$

Accordingly, we have that

$$\begin{aligned}
& \lim_{r \rightarrow +0} r|B_k(r)| \\
&= \lim_{r \rightarrow +0} r(i_r^{(k-1)} - i_r^{(k)}) \geq \lim_{r \rightarrow +0} r(i_r^{(k-1)} - i_r^*) \\
&\geq \lim_{r \rightarrow +0} \left( 1 - \sum_{(b_k, \dots, b_1) \in \Xi_k \setminus \{\mathbf{e}\}} (f^{b_{k-1}}(B_{k-1}(r)) \cdots f^{b_1}(B_1(r))) - r \right) \\
&= 1 - (1 - \lambda_k) = \lambda_k,
\end{aligned}$$

and thus  $\lim_{r \rightarrow +0} r|B_k(r)| = \lambda_k$  is obtained.

Now, we have shown (6.1) and (6.2) for every  $k \in \{1, 2, \dots, m\}$ , which implies

$$\begin{aligned}
& \lim_{r \rightarrow +0} P_r^{(\text{win})}(m) \\
&= \lim_{r \rightarrow +0} \sum_{k=1}^m \left( \left( \prod_{i \in B_k(r) \cup \dots \cup B_1(r)} q \right) \left( \sum_{(b_k, \dots, b_1) \in \Xi_k} (f^{b_k}(B_k(r)) \cdots f^{b_1}(B_1(r))) \right) \right) \\
&= \lim_{r \rightarrow +0} \sum_{k=1}^m \left( \left( \prod_{k'=1}^k q^{|B_{k'}(r)|} \right) \left( \sum_{(b_k, \dots, b_1) \in \Xi_k} (f^{b_k}(B_k(r)) \cdots f^{b_1}(B_1(r))) \right) \right) \\
&= \sum_{k=1}^m \left( \left( \prod_{k'=1}^k e^{-\lambda_{k'}} \right) \left( \sum_{(b_k, \dots, b_1) \in \Xi_k} \frac{\lambda_k^{b_k}}{b_k!} \frac{\lambda_{k-1}^{b_{k-1}}}{b_{k-1}!} \cdots \frac{\lambda_1^{b_1}}{b_1!} \right) \right) = \sum_{k=1}^m e^{-\sum_{k'=1}^k \lambda_{k'}},
\end{aligned}$$

where the last equality is obtained from (4.1).  $\square$



**7. Secretary Problem.** In this section, we show that an optimal strategy for secretary problem attains our lower bounds obtained in Theorem 4.3. We discuss a sequence of 0/1 random variables  $X_2, X_3, \dots, X_N$  satisfying  $\Pr[X_i = 1] = 1/i$ , for any  $i \in \{2, 3, \dots, N\}$ . In the following,  $q_i$  denotes the probability of failure  $1 - 1/i$  and  $r_i$  denotes the odds  $1/(i - 1)$  of  $X_i$ .

Gilbert and Mosteller [13] showed that an optimal strategy for secretary problem with  $m$ -stoppings is attained by a threshold strategy. We denote an optimal threshold strategy by  $\text{Threshold}(i_N^{(m)}, i_N^{(m-1)}, \dots, i_N^{(1)})$ , which is dependent on the length  $N$ . We also introduce a block partition  $\{B_{m+1}(N), B_m(N), \dots, B_1(N)\}$  of index set  $\{2, 3, \dots, N\}$  defined by

$$B_k(N) = \begin{cases} \{i \in \{2, 3, \dots, N\} \mid i_N^{(1)} \leq i \leq N\} & (k = 1), \\ \{i \in \{2, 3, \dots, N\} \mid i_N^{(k)} \leq i < i_N^{(k-1)}\} & (1 < k \leq m), \\ \{i \in \{2, 3, \dots, N\} \mid 2 \leq i < i_N^{(m)}\} & (k = m + 1). \end{cases}$$

LEMMA 7.1. *Let  $(\lambda_1, \lambda_2, \dots, \lambda_m)$  be a vector consisting of all positive entries. Assume that a threshold strategy  $\text{Threshold}(i_N^{(m)}, i_N^{(m-1)}, \dots, i_N^{(1)})$  and a corresponding partition  $\{B_{m+1}(N), B_m(N), \dots, B_1(N)\}$  satisfy conditions*

$$(7.1) \quad \lim_{N \rightarrow \infty} i_N^{(k)} = +\infty \quad \text{and} \quad \lim_{N \rightarrow \infty} \sum_{i \in B_k(N)} r_i = \lambda_k,$$

for each  $k \in \{1, 2, \dots, m\}$ . Then the partition also satisfies

$$(7.2) \quad \lim_{N \rightarrow \infty} \prod_{i \in B_k(N)} q_i = e^{-\lambda_k} \quad \text{and} \quad \forall b \in \{0, 1, \dots, m\}, \quad \lim_{N \rightarrow \infty} f^b(B_k(N)) = \frac{\lambda_k^b}{b!},$$

for each  $k \in \{1, 2, \dots, m\}$ .

PROOF. Condition (7.1) directly implies  $\lim_{N \rightarrow \infty} r_{i_N^{(k)}} = 0$ . It is easy to see that

$$(7.3) \quad \begin{aligned} \ln \prod_{i \in B_k(N)} q_i &= \sum_{i \in B_k(N)} \ln(1 - 1/i) = \sum_{i \in B_k(N)} (\ln(i - 1) - \ln i) \\ &= \ln(i_N^{(k)} - 1) - \ln(i_N^{(k-1)} - 1) = \ln \frac{i_N^{(k)} - 1}{i_N^{(k-1)} - 1} \end{aligned}$$

and

$$\sum_{i=i_N^{(k)}-1}^{i_N^{(k-1)}-2} \frac{1}{i} \geq \ln(i_N^{(k-1)} - 1) - \ln(i_N^{(k)} - 1) \geq \sum_{i=i_N^{(k)}}^{i_N^{(k-1)}-2} \frac{1}{i},$$

where  $i_N^{(0)}$  denotes  $N + 1$ . From the above, we obtain that

$$\begin{aligned} \lim_{N \rightarrow \infty} \sum_{i=i_N^{(k)}}^{i_N^{(k-1)}-1} r_i &\geq - \lim_{N \rightarrow \infty} \ln \prod_{i \in B_k(N)} q_i \geq \lim_{N \rightarrow \infty} \sum_{i=i_N^{(k)}+1}^{i_N^{(k-1)}-1} r_i \\ \lim_{N \rightarrow \infty} \sum_{i \in B_k(N)} r_i &\geq - \lim_{N \rightarrow \infty} \ln \prod_{i \in B_k(N)} q_i \geq \lim_{N \rightarrow \infty} \left( -r_{i_N^{(k)}} + \sum_{i \in B_k(N)} r_i \right) \\ -\lambda_k &\leq \lim_{N \rightarrow \infty} \ln \prod_{i \in B_k(N)} q_i \leq -\lambda_k. \end{aligned}$$

Accordingly, we have  $\lim_{N \rightarrow \infty} \prod_{i \in B_k(N)} q_i = e^{-\lambda_k}$ .

We omit the case of  $f^0(B_k(N))$  in the following, since equality  $f^0(B_k(N)) = 1 = \frac{\lambda_k^0}{0!}$  holds permanently. We discuss cases  $b \in \{1, 2, \dots, m\}$ . The size of block  $B_k(N)$  satisfies the following;

$$\lim_{N \rightarrow \infty} |B_k(N)| r_{i_N^{(k)}} \geq \lim_{N \rightarrow \infty} \sum_{i=i_N^{(k)}}^{i_N^{(k-1)}-1} r_i = \lim_{N \rightarrow \infty} \sum_{i \in B_k(N)} r_i = \lambda_k > 0.$$

The positivity of  $\lambda_k$  implies  $\lim_{N \rightarrow \infty} |B_k(N)| = +\infty$ . There exists an integer  $N'$  such that for any integer  $N > N'$ , the size of  $B_k(N)$  exceeds  $m$  and thus

$$\forall b \in \{1, 2, \dots, m\}, \quad f^b(B_k(N)) = \sum_{B' \subseteq B_k(N), |B'|=b} \left( \prod_{i \in B'} r_i \right).$$

Now, we show  $\lim_{N \rightarrow \infty} f^b(B_k(N)) = \frac{\lambda_k^b}{b!}$  by induction on  $b$ . Condition (7.1) directly implies the case  $b = 1$ , i.e., the equality

$$\lim_{N \rightarrow \infty} f^1(B_k(N)) = \lim_{N \rightarrow \infty} \sum_{i \in B_k(N)} r_i = \frac{\lambda_k^1}{1!}$$

holds. Next, assume that  $\lim_{N \rightarrow \infty} f^{b-1}(B_k(N)) = \frac{\lambda_k^{b-1}}{(b-1)!}$ . Then, it is easy to see that

$$\begin{aligned} &\lim_{N \rightarrow \infty} f^b(B_k(N)) \\ &= \lim_{N \rightarrow \infty} \sum_{B' \subseteq B_k(N), |B'|=b} \left( \prod_{i \in B'} r_i \right) \leq \lim_{N \rightarrow \infty} \frac{\left( \sum_{i \in B_k(N)} r_i \right)^b}{b!} = \frac{\lambda_k^b}{b!}. \end{aligned}$$

The induction hypothesis on  $b$  implies that

$$\begin{aligned} \lim_{N \rightarrow \infty} f^b(B_k(N)) &= \lim_{N \rightarrow \infty} \sum_{B' \subseteq B_k(N), |B'|=b} \left( \prod_{i \in B'} r_i \right) \\ &\geq \lim_{N \rightarrow \infty} \frac{f^{b-1}(B_k(N)) \left( \sum_{i \in B_k(N)} r_i - b r_{i_N^{(k)}} \right)}{b} = \frac{\lambda_k^{b-1}}{(b-1)!} \frac{\lambda_k}{b} = \frac{\lambda_k^b}{b!}. \end{aligned}$$

Thus, we have shown  $\lim_{N \rightarrow \infty} f^b(B_k(N)) = \frac{\lambda_k^b}{b!}$ .  $\square$

Following theorem gives the win probability of secretary problem.

**THEOREM 7.2.** *Let  $(\lambda_1, \lambda_2, \dots, \lambda_m)$  be a unique solution of (4.1). Given a sequence of 0/1 random variables  $X_2, X_3, \dots, X_N$  satisfying  $\Pr[X_i = 1] = 1/i$  ( $\forall i \in \{2, 3, \dots, N\}$ ), the probability of win  $P_N^{(\text{win})}(m)$  of an optimal strategy for secretary problem with  $m$ -stoppings defined on  $X_2, X_3, \dots, X_N$  satisfies*

$$\lim_{N \rightarrow \infty} P_N^{(\text{win})}(m) = \sum_{k=1}^m e^{-\sum_{k'=1}^k \lambda_{k'}}.$$

**PROOF.** It is well-known that an optimal strategy for secretary problem with  $m$ -stoppings is attained by an optimal threshold strategy  $\text{Threshold}(i_N^{(m)}, i_N^{(m-1)}, \dots, i_N^{(1)})$  satisfying that for any  $k \in \{1, 2, \dots, m\}$ ,  $\text{Threshold}(i_N^{(k)}, i_N^{(k-1)}, \dots, i_N^{(1)})$  is also an optimal strategy for  $k$ -stopping problem.

Here, we show property (7.1) in Lemma 7.1 by induction on  $k$  (which implies (7.2), simultaneously). When  $k = 1$ ,  $\text{Threshold}(i_N^{(1)})$  is optimal to single-stopping problem. A classical theorem of secretary problem (see [10] and/or Sum the Odds Theorem [6]) implies that when  $N$  is sufficiently large,

$$(7.4) \quad 1 \leq \sum_{i=i_N^{(1)}}^N r_i \leq 1 + r_{i_N^{(1)}},$$

and thus

$$1 \geq \lim_{N \rightarrow \infty} \left( -r_{i_N^{(1)}} + \sum_{i=i_N^{(1)}}^N r_i \right) = \lim_{N \rightarrow \infty} \sum_{i=i_N^{(1)}+1}^N r_i = \lim_{N \rightarrow \infty} \sum_{i=i_N^{(1)}}^{N-1} \frac{1}{i} \geq \lim_{N \rightarrow \infty} \ln \frac{N}{i_N^{(1)}}.$$

From the above, we obtain  $\lim_{N \rightarrow \infty} i_N^{(1)} = +\infty$ . Inequalities (7.4) directly implies

$$\begin{aligned} \lim_{N \rightarrow \infty} 1 &\leq \lim_{N \rightarrow \infty} \sum_{i=i_N^{(1)}}^N r_i \leq \lim_{N \rightarrow \infty} \left(1 + r_{i_N^{(1)}}\right) \\ 1 &\leq \lim_{N \rightarrow \infty} \sum_{i=i_N^{(1)}}^N r_i \leq \lim_{N \rightarrow \infty} \left(1 + \frac{1}{i_N^{(1)} - 1}\right) = 1 \end{aligned}$$

and thus  $\lim_{N \rightarrow \infty} \sum_{i \in B_1(N)} r_i = \lim_{N \rightarrow \infty} \sum_{i=i_N^{(1)}}^N r_i = 1 = \lambda_1$ . We have shown property (7.1) and obtained (7.2), when  $k = 1$ .

Now, we describe  $k$ -th induction step (where  $k \leq m$ ) under the assumption that for any  $k' \in \{1, 2, \dots, k-1\}$ ,

$$(7.5) \quad \lim_{N \rightarrow \infty} i_N^{(k')} = +\infty \quad \text{and} \quad \forall b \in \{0, 1, \dots, m\}, \quad \lim_{N \rightarrow \infty} f^b(B_{k'}(N)) = \frac{\lambda_{k'}^b}{b!}.$$

Let  $\mathbf{e}$  be a unit  $k$ -vector  $(1, 0, 0, \dots, 0)$ . Lemma 4.1 says that  $\mathbf{e} \in \Xi_k$  and every vector  $(b_k, b_{k-1}, \dots, b_1) \in \Xi_k \setminus \{\mathbf{e}\}$  satisfies  $b_k = 0$ . Obviously, we have

$$\begin{aligned} &\sum_{(b_k, \dots, b_1) \in \Xi_k} \left( f^{b_k}(B_k(N)) f^{b_{k-1}}(B_{k-1}(N)) \cdots f^{b_1}(B_1(N)) \right) \\ &= \sum_{i \in B_k(N)} r_i + \sum_{(b_k, \dots, b_1) \in \Xi_k \setminus \{\mathbf{e}\}} \left( f^{b_{k-1}}(B_{k-1}(N)) f^{b_{k-2}}(B_{k-2}(N)) \cdots f^{b_1}(B_1(N)) \right). \end{aligned}$$

Similarly to the proof of equality (6.5), the induction hypothesis (7.5) and the definition of equality system (4.1) imply

$$\lim_{N \rightarrow \infty} \left( \sum_{(b_k, \dots, b_1) \in \Xi_k \setminus \{\mathbf{e}\}} \left( f^{b_{k-1}}(B_{k-1}(N)) f^{b_{k-2}}(B_{k-2}(N)) \cdots f^{b_1}(B_1(N)) \right) \right) = 1 - \lambda_k.$$

Now we introduce a threshold strategy  $\text{Threshold}(i, i_N^{(k-1)}, i_N^{(k-2)}, \dots, i_N^{(1)})$  ( $\forall i \in \{1, 2, \dots, i_N^{(k-1)}\}$ ) for  $k$ -stopping problem and employ a one-stage look-ahead approach [1]. We denote a probability of win of threshold strategy  $\text{Threshold}(i, i_N^{(k-1)}, i_N^{(k-2)}, \dots, i_N^{(1)})$  by  $P_N^{(\text{win})}(k, i)$ . A difference of a pair of

win probabilities  $P_N^{(\text{win})}(k, i-1)$  and  $P_N^{(\text{win})}(k, i)$  satisfies

$$\begin{aligned}
 & P_N^{(\text{win})}(k, i-1) - P_N^{(\text{win})}(k, i) \\
 &= \left( \prod_{i'=i-1}^N q_{i'} \right) \left( \sum_{i'=i-1}^{i_N^{(k-1)}-1} r_{i'} + \sum_{(b_k, \dots, b_1) \in \Xi_k \setminus \{e\}} (f^{b_{k-1}}(B_{k-1}(N)) \cdots f^{b_1}(B_1(N))) \right) \\
 &\quad - \left( \prod_{i'=i}^N q_{i'} \right) \left( \sum_{i'=i}^{i_N^{(k-1)}-1} r_{i'} + \sum_{(b_k, \dots, b_1) \in \Xi_k \setminus \{e\}} (f^{b_{k-1}}(B_{k-1}(N)) \cdots f^{b_1}(B_1(N))) \right) \\
 &= \left( \prod_{i'=i}^N q_{i'} \right) \left( \begin{aligned} & (q_{i-1} - 1) \sum_{i'=i}^{i_N^{(k-1)}-1} r_{i'} + q_{i-1} r_{i-1} \\ & + (q_{i-1} - 1) \sum_{(b_k, \dots, b_1) \in \Xi_k \setminus \{e\}} (f^{b_{k-1}}(B_{k-1}(N)) \cdots f^{b_1}(B_1(N))) \end{aligned} \right) \\
 &= \left( \prod_{i'=i}^N q_{i'} \right) (1 - q_{i-1}) \left( \begin{aligned} & - \sum_{i'=i}^{i_N^{(k-1)}-1} r_{i'} + 1 \\ & - \sum_{(b_k, \dots, b_1) \in \Xi_k \setminus \{e\}} (f^{b_{k-1}}(B_{k-1}(N)) \cdots f^{b_1}(B_1(N))) \end{aligned} \right).
 \end{aligned}$$

From the above, it is easy to see that  $P_N^{(\text{win})}(k, i-1) - P_N^{(\text{win})}(k, i)$  is strictly increasing with  $i$  and a sequence of win probabilities

$$(P_N^{(\text{win})}(k, 2), P_N^{(\text{win})}(k, 3), \dots, P_N^{(\text{win})}(k, i_N^{(k-1)}))$$

is unimodal. In a similar way with a discussion about (6.6) in the proof of Theorem 6.2, we can show that there exists a large integer  $N'$  satisfying

$$\forall N > N', \quad P_N^{(\text{win})}(k, i_N^{(k-1)} - 1) - P_N^{(\text{win})}(k, i_N^{(k-1)}) > 0.$$

Next, we consider the pair  $P_N^{(\text{win})}(k, 2)$  and  $P_N^{(\text{win})}(k, 3)$ . The induction hypothesis (7.5) implies  $\lim_{N \rightarrow \infty} \sum_{i=3}^{i_N^{(k-1)}-1} r_i = +\infty$ . Consequently, it is clear that

$$\lim_{N \rightarrow \infty} \left( - \sum_{i'=3}^{i_N^{(k-1)}-1} r_{i'} + 1 - \sum_{(b_k, \dots, b_1) \in \Xi_k \setminus \{e\}} (f^{b_{k-1}}(B_{k-1}(N)) \cdots f^{b_1}(B_1(N))) \right) = -\infty$$

and thus there exists a large integer  $N''$  satisfying

$$\forall N > N'', \quad P_N^{(\text{win})}(k, 2) - P_N^{(\text{win})}(k, 3) < 0.$$

From the above, we have shown that  $\forall N > \max\{N', N''\}$ ,

$$P_N^{(\text{win})}(k, 2) < P_N^{(\text{win})}(k, 3) \text{ and } P_N^{(\text{win})}(k, i_N^{(k-1)} - 1) > P_N^{(\text{win})}(k, i_N^{(k-1)}),$$

which implies that there exists an index  $i_N^* \in \{4, 5, \dots, i_N^{(k-1)} - 1\}$  satisfying

$$\begin{aligned} P_N^{(\text{win})}(k, 2) &< P_N^{(\text{win})}(k, 3) \\ &< \dots < P_N^{(\text{win})}(k, i_N^* - 2) < P_N^{(\text{win})}(k, i_N^* - 1) \geq P_N^{(\text{win})}(k, i_N^*) \\ &> \dots > P_N^{(\text{win})}(k, i_N^{(k-1)}). \end{aligned}$$

An optimal threshold strategy is attained by either **Threshold**( $i_N^*, i_N^{(k-1)}, \dots, i_N^{(1)}$ ) or **Threshold**( $i_N^* - 1, i_N^{(k-1)}, \dots, i_N^{(1)}$ ) and thus  $i_N^* - 1 \leq i_N^{(k)} \leq i_N^*$ . Clearly from the inequality  $P_N^{(\text{win})}(k, i_N^* - 1) \geq P_N^{(\text{win})}(k, i_N^*)$ , index  $i_N^*$  satisfies

$$\sum_{i'=i_N^*}^{i_N^{(k-1)}-1} r_{i'} \leq 1 - \sum_{(b_k, \dots, b_1) \in \Xi_k \setminus \{\mathbf{e}\}} (f^{b_{k-1}}(B_{k-1}(N)) \cdots f^{b_1}(B_1(N))).$$

From the above, we obtain

$$\begin{aligned} \lambda_k &= \lim_{N \rightarrow \infty} \left( 1 - \sum_{(b_k, \dots, b_1) \in \Xi_k \setminus \{\mathbf{e}\}} (f^{b_{k-1}}(B_{k-1}(N)) \cdots f^{b_1}(B_1(N))) \right) \\ &\geq \lim_{N \rightarrow \infty} \left( \sum_{i'=i_N^*}^{i_N^{(k-1)}-1} r_{i'} \right) = \lim_{N \rightarrow \infty} \left( \sum_{i'=i_N^*-1}^{i_N^{(k-1)}-2} \frac{1}{i'} \right) \geq \lim_{N \rightarrow \infty} \left( \ln \frac{i_N^{(k-1)} - 1}{i_N^* - 1} \right). \end{aligned}$$

The above inequality, the induction hypothesis (7.5) ( $\lim_{N \rightarrow \infty} i_N^{(k-1)} = +\infty$ ) and inequality  $i_N^{(k)} \geq i_N^* - 1$  directly imply

$$\lim_{N \rightarrow \infty} i_N^* = +\infty \quad \text{and} \quad \lim_{N \rightarrow \infty} i_N^{(k)} = +\infty.$$

Additionally, we have that

$$\begin{aligned} \lim_{N \rightarrow \infty} \sum_{i' \in B_k(N)} r_{i'} &\leq \lim_{N \rightarrow \infty} \sum_{i'=i_N^*-1}^{i_N^{(k-1)}-1} r_{i'} = \lim_{N \rightarrow \infty} \left( r_{i_N^*-1} + \sum_{i'=i_N^*}^{i_N^{(k-1)}-1} r_{i'} \right) \\ &\leq \lim_{N \rightarrow \infty} \left( \frac{1}{i_N^* - 2} + 1 - \sum_{(b_k, \dots, b_1) \in \Xi_k \setminus \{\mathbf{e}\}} (f^{b_{k-1}}(B_{k-1}(N)) \cdots f^{b_1}(B_1(N))) \right) \\ &= 1 - (1 - \lambda_k) = \lambda_k. \end{aligned}$$

Since  $P_N^{(\text{win})}(k, i_N^* - 2) < P_N^{(\text{win})}(k, i_N^* - 1)$ , index  $i_N^*$  satisfies

$$\sum_{i'=i_N^*-1}^{i_N^{(k-1)}-1} r_{i'} > 1 - \sum_{(b_k, \dots, b_1) \in \Xi_k \setminus \{\mathbf{e}\}} (f^{b_{k-1}}(B_{k-1}(N)) \cdots f^{b_1}(B_1(N))).$$

Consequently, we have

$$\begin{aligned} \lim_{N \rightarrow \infty} \sum_{i' \in B_k(N)} r_{i'} &\geq \lim_{N \rightarrow \infty} \sum_{i'=i_N^*}^{i_N^{(k-1)}-1} r_{i'} = \lim_{N \rightarrow \infty} \left( -r_{i_N^*-1} + \sum_{i'=i_N^*-1}^{i_N^{(k-1)}-1} r_{i'} \right) \\ &\geq \lim_{N \rightarrow \infty} \left( \frac{-1}{i_N^*-2} + 1 - \sum_{(b_k, \dots, b_1) \in \Xi_k \setminus \{\mathbf{e}\}} (f^{b_{k-1}}(B_{k-1}(N)) \cdots f^{b_1}(B_1(N))) \right) \\ &= 1 - (1 - \lambda_k) = \lambda_k. \end{aligned}$$

From the above, we obtain  $\lim_{N \rightarrow \infty} \sum_{i \in B_k(N)} r_i = \lambda_k$ . Now, we have shown (7.1) and (7.2) for each  $k \in \{1, 2, \dots, m\}$ .

The probability of win  $P_N^{(\text{win})}(m)$  satisfies

$$\begin{aligned} \lim_{N \rightarrow \infty} P_N^{(\text{win})}(m) &= \lim_{N \rightarrow \infty} \sum_{k=1}^m \left( \left( \prod_{i \in B_k(N) \cup \dots \cup B_1(N)} q_i \right) \left( \sum_{(b_k, \dots, b_1) \in \Xi_k} (f^{b_k}(B_k(N)) \cdots f^{b_1}(B_1(N))) \right) \right) \\ &= \sum_{k=1}^m \left( \left( \prod_{k'=1}^k e^{-\lambda_{k'}} \right) \left( \sum_{(b_k, \dots, b_1) \in \Xi_k} \frac{\lambda_k^{b_k}}{b_k!} \frac{\lambda_{k-1}^{b_{k-1}}}{b_{k-1}!} \cdots \frac{\lambda_1^{b_1}}{b_1!} \right) \right) = \sum_{k=1}^m e^{-\sum_{k'=1}^k \lambda_{k'}}, \end{aligned}$$

where the last equality is obtained from (4.1).  $\square$

Lastly, we show a relation between threshold values and the probability of win of secretary problem indicated by Gilbert and Mosteller [13].

**THEOREM 7.3.** *Given a sequence of 0/1 random variables  $X_2, X_3, \dots, X_N$  satisfying  $\Pr[X_i = 1] = 1/i$  ( $\forall i \in \{2, 3, \dots, N\}$ ), an optimal (threshold) strategy  $\text{Threshold}(i_N^{(m)}, i_N^{(m-1)}, \dots, i_N^{(1)})$  for secretary problem with  $m$ -stoppings satisfies*

$$\lim_{N \rightarrow \infty} \sum_{k=1}^m \frac{i_N^{(k)}}{N} = \lim_{N \rightarrow \infty} P_N^{(\text{win})}(m)$$

where  $P_N^{(\text{win})}(m)$  denotes a corresponding probability of win.

PROOF. In the following,  $i_N^{(0)}$  denotes  $N + 1$  and  $(\lambda_1, \lambda_2, \dots, \lambda_m)$  denotes a unique solution of (4.1). Properties (7.1) and (7.2) shown in the proof of Theorem 7.2, and equality (7.3) imply that for any  $k \in \{1, 2, \dots, m\}$ ,

$$\begin{aligned} \lim_{N \rightarrow \infty} \ln \frac{i_N^{(k)}}{i_N^{(k-1)}} &= \lim_{N \rightarrow \infty} \ln \left( \frac{i_N^{(k)} - 1}{i_N^{(k-1)} - 1} \frac{i_N^{(k-1)} - 1}{i_N^{(k-1)}} \frac{i_N^{(k)}}{i_N^{(k)} - 1} \right) \\ &= \lim_{N \rightarrow \infty} \left( \ln \left( \prod_{i \in B_k(N)} q_i \right) + \ln \left( 1 - \frac{1}{i_N^{(k-1)}} \right) + \ln \left( 1 + \frac{1}{i_N^{(k)} - 1} \right) \right) = -\lambda_k. \end{aligned}$$

Thus, we have

$$\begin{aligned} \lim_{N \rightarrow \infty} \ln \frac{i_N^{(k)}}{N} &= \lim_{N \rightarrow \infty} \ln \left( \left( \frac{i_N^{(k)}}{i_N^{(k-1)}} \right) \left( \frac{i_N^{(k-1)}}{i_N^{(k-2)}} \right) \cdots \left( \frac{i_N^{(1)}}{i_N^{(0)}} \right) \left( \frac{N+1}{N} \right) \right) \\ &= \lim_{N \rightarrow \infty} \left( \sum_{k'=1}^k \ln \left( \frac{i_N^{(k')}}{i_N^{(k'-1)}} \right) + \ln \left( \frac{N+1}{N} \right) \right) = -\sum_{k'=1}^k \lambda_{k'} \end{aligned}$$

and

$$\lim_{N \rightarrow \infty} \frac{i_N^{(k)}}{N} = e^{-\sum_{k'=1}^k \lambda_{k'}}.$$

The above equality and Theorem 7.2 implies

$$\lim_{N \rightarrow \infty} \sum_{k=1}^m \frac{i_N^{(k)}}{N} = \sum_{k=1}^m e^{-\sum_{k'=1}^k \lambda_{k'}} = \lim_{N \rightarrow \infty} P_N^{(\text{win})}(m).$$

□

**Acknowledgments.** The authors thank Professor A. V. Gnedin for his comment on using a Poisson approximation to find the asymptotic probability of win at the 34th Conference on Stochastic Processes and their Applications, Osaka.

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